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Local controllability to stationary trajectories of a one-dimensional simplified model arising in turbulence

Sorin Micu*†, Takéo Takahashi‡

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Contents

Abstract

This article studies the local controllability to trajectories of a one dimensional model for turbulence. By linearization we are led to an equation with a non local term whose controllability properties are analyzed by using Fourier decomposition and biorthogonal techniques. Once the existence of controls is proved and the dependence of their norms with respect to the time is established for the linearized model, a fixed point method allows us to deduce the result for the nonlinear initial problem.

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1 Introduction

In [7], the authors tackle the local null controllability of a Ladyzhenskaya-Smagorinsky model of turbulence. More precisely, they consider the following control problem

$$\begin{cases} \partial_t w + (w\cdot\nabla)w - \nu(\|\nabla w\|^2_{L^2(\Omega)})\Delta w + \nabla q = u\chi_\omega \quad \text{in } (0,T)\times\Omega, \\ \operatorname{div} w = 0 \quad \text{in } (0,T)\times\Omega, \\ w = 0 \quad \text{on } (0,T)\times\partial\Omega \\ w(0,\cdot) = w_0 \quad \text{in } \Omega, \end{cases}$$

where $\nu : \mathbb{R}_+ \to [\nu_0,\infty)$ is a C^1 function, with $\nu_0 > 0$ and with bounded derivatives. Following the method used for the controllability of the Navier-Stokes system (see [6]), they show a Carleman estimate for the linearized system and, with a fixed point argument, they obtain the local controllability to the null state ($w(T,\cdot)\equiv 0$). In Section 6 of [7], it is mentioned that at the contrary to the Navier-Stokes system, it is not clear how to obtain the local controllability to the trajectories. Indeed, in the linearized system, one has to deal with non-local terms.

In [8], a partial answer to the above question is given: the authors consider the linear heat and the linear wave system and show that one can recover the controllability properties of these systems if some non local terms are added. More precisely, their results are obtained for any dimension in space, but they need that the nonlocal integral terms are analytic in space.

Unhappily, in the above paper, the proof is given through a contradiction argument and in particular, it is not clear how to keep the cost of the linear heat equation in order to tackle nonlinear problems. Our aim is here to study the controllability of a Burgers system with variable viscosity. This can be seen as a simplified model for the turbulence model considered in [7]. More precisely, our system writes as

$$\begin{cases} \partial_t w - \nu(\|\partial_x w\|^2_{L^2(0,\pi)})\partial_{xx} w + w\partial_x w = f^S + u\chi_\omega \quad \text{in } (0,T)\times(0,\pi), \\ w(t,0) = w(t,\pi) = 0 \quad t\in(0,T), \\ w(0,\cdot) = w_0 := y_0 + w^S \quad \text{in } (0,\pi), \end{cases} \tag{1.1}$$

where ω is a non empty open interval of $(0,\pi)$. In what follows, we assume that $\nu : \mathbb{R}_+ \to [\nu_0,\infty)$ is a C^2 function, with $\nu_0 > 0$. In (1.1) the function f^S depends only on x and belongs to $L^2(0,\pi)$.

Our aim in this paper is to obtain the local exact controllability to the stationary trajectories. More precisely, w^S and f^S in the above system are related through the stationary equations:

$$\begin{cases} -\nu(\|\partial_x w^S\|^2_{L^2(0,\pi)})\partial_{xx} w^S + w^S\partial_x w^S = f^S \quad \text{in } (0,T)\times(0,\pi), \\ w^S(0) = w^S(\pi) = 0. \end{cases} \tag{1.2}$$

We can write the above problem as a null controllability problem by setting

$$y := w - w^S.$$

Then y satisfies the following nonlinear heat-type system:

$$\begin{cases} \partial_t y - \nu^S\partial_{xx} y - \mu\left(\int_0^\pi (\partial_x w^S)(\partial_x y)\,dx\right)\partial_{xx} w^S + w^S\partial_x y + y\partial_x w^S = F + u\chi_\omega, \\ y(t,0) = y(t,\pi) = 0, \\ y(0,\cdot) = y_0, \end{cases}$$

and

$$F = -y\partial_x y + \left[\nu(\|\partial_x y + \partial_x w^S\|^2_{L^2(0,\pi)}) - \nu(\|\partial_x w^S\|^2_{L^2(0,\pi)})\right]\partial_{xx} y \\ + \Bigg[\nu(\|\partial_x y + \partial_x w^S\|^2_{L^2(0,\pi)}) - \nu(\|\partial_x w^S\|^2_{L^2(0,\pi)}) \\ - 2\nu'(\|\partial_x w^S\|^2_{L^2(0,\pi)})\left(\int_0^\pi (\partial_x w^S)(\partial_x y)\ dx\right)\Bigg]\partial_{xx} w^S, \tag{1.3}$$

with

$$\nu^S := \nu(\|\partial_x w^S\|^2_{L^2(0,\pi)}) > 0 \quad \text{and} \quad \mu = 2\nu'(\|\partial_x w^S\|^2_{L^2(0,\pi)}).$$

By integrating by parts and linearization we are lead to study

$$\begin{cases} \partial_t y - \partial_{xx} y + \mu\left(\int_0^\pi ay\ dx\right) a + w^S\partial_x y + y\partial_x w^S = u\chi_\omega, \\ y(t,0) = y(t,\pi) = 0, \\ y(0,\cdot) = y_0, \end{cases} \tag{1.4}$$

where a is smooth function and $\mu \in \mathbb{R}$. We have assumed $\nu^S = 1$ to simplify.

Let us consider

$$W(x) := \frac{1}{2}\int_0^x w^S\ ds$$

and let us set

$$z(t,x) := y(t,x)e^{-W(x)}. \tag{1.5}$$

Then standard calculation yields

$$\partial_t y - \partial_{xx} y + \mu\left(\int_0^\pi ay\ dx\right) a + w^S\partial_x y + y\partial_x w^S \\ = e^W\Bigg(\partial_t z - \partial_{xx} z - 2W'(\partial_x z) - W''z - (W')^2 z \\ + \mu\left(\int_0^\pi ae^W z\ dx\right) ae^{-W} + w^S(\partial_x z + W'z) + z\partial_x w^S\Bigg).$$

The system (1.4) is transformed into

$$\begin{cases} \partial_t z - \partial_{xx} z + \int_0^\pi \mathcal{K}(\xi,\cdot)z(\xi)\ d\xi + pz = v\chi_\omega, \\ z(t,0) = z(t,\pi) = 0, \\ z(0,\cdot) = z_0, \end{cases} \tag{1.6}$$

$$p = \left(\frac{1}{2}\partial_x w^S + \frac{1}{4}(w^S)^2\right),$$
$$\mathcal{K}(\xi,x) = a(\xi)e^{W(\xi)}a(x)e^{-W(x)}.$$

Let us define the unbounded operator $(D(A),A)$ in $L^2(0,\pi)$ as follows

$$\begin{aligned} &D(A) = H^2(0,\pi) \cap H^1_0(0,\pi), \\ &Az = -\partial_{xx} z + \int_0^\pi \mathcal{K}(\xi,\cdot)z(\xi)\,\mathrm{d}\xi + pz \qquad (z \in D(A)), \end{aligned} \tag{1.7}$$

where $p \in L^\infty(0,\pi)$ and the kernel $\mathcal{K} \in L^2\left((0,\pi)^2\right)$.

With this notation, system (1.6) can be equivalently written as follows

$$
\begin{cases} \partial_t z(t) + Az(t) = v\chi_\omega, \\ z(0) = z_0. \end{cases} \tag{1.8}
$$

The adjoint $(D(A^*), A^*)$ of the operator $(D(A), A)$ is given by

$$
\begin{aligned}
&D(A^*) = H^2(0,\pi) \cap H_0^1(0,\pi), \\
&A^* w = -\partial_{xx} w + \int_0^\pi K(\,\cdot\,,\xi) w(\xi)\,\mathrm{d}\xi + qw \qquad (w \in D(A^*)),
\end{aligned} \tag{1.9}
$$

where $K \in L^2\left((0,\pi)^2\right)$ and $q \in L^\infty(0,\pi)$ are given by

$$
K(x,\xi) = \overline{\mathcal{K}}(x,\xi), \quad q = \overline{p}.
$$

Also, we introduce the following "adjoint problem":

$$
\begin{cases} \partial_t \varphi(t) + A^*\varphi(t) = 0, \\ \varphi(0) = \varphi_0. \end{cases} \tag{1.10}
$$

The following result is classic in control theory (see [5] or Theorem 11.2.1 in [13]).

Theorem 1.1. *Let $T > 0$ and let $\omega \subset [0,\pi]$ be a non empty open set. For each $z_0 \in L^2(0,\pi)$ there exists $v \in L^2((0,T)\times\omega)$ such that the solution z of* (1.8) *vanishes at T, if and only if there exists a constant $C = C(T)$ such that the following inequality holds*

$$
\|\varphi(T)\|^2_{L^2(0,\pi)} \leqslant C \int_0^T \int_\omega |\varphi(t,x)|^2\,\mathrm{d}x\,\mathrm{d}t, \tag{1.11}
$$

for any $\varphi_0 \in L^2(0,\pi)$ and φ solution of (1.10)*. Moreover, if* (1.11) *holds, then there exists a control $v \in L^2(0,T)$ which verifies*

$$
\|v\|^2_{L^2(0,T)} \leqslant C\|z_0\|^2_{L^2(0,\pi)} \qquad (z_0 \in L^2(0,\pi)). \tag{1.12}
$$

A key result of this paper is the following theorem concerning the observability inequality (1.11) which also estimates the behavior of the constant C as T tends to zero.

Theorem 1.2. *Let $\omega \subset [0,\pi]$ be a non empty open set. Suppose that the kernel K is degenerate, i.e. there exist two functions $\alpha, \beta \in L^2(0,\pi)$ such that $K(x,\xi) = \alpha(x)\beta(\xi)$ for each $(x,\xi) \in [0,\pi]^2$ and suppose that α is not identically zero in ω. Then, there exist three positive constants T_0, M_0 and ς such that, for any $T \in (0,T_0)$ and $\varphi_0 \in L^2(0,\pi)$, the corresponding solution φ of equation* (1.10) *verifies the observability inequality*

$$
\|\varphi(T)\|^2_{L^2(0,\pi)} \leqslant M_0 \exp\left(\frac{\varsigma}{T}\right) \int_0^T \int_\omega |\varphi(t,x)|^2\,\mathrm{d}x\,\mathrm{d}t. \tag{1.13}
$$

To prove Theorem 1.2 we adopt the following strategy:

- We show that there exists a Riesz basis of $L^2(0,\pi)$ formed by generalized eigenvectors of the operator $(D(A^*), A^*)$.

  In order to do this we analyze in Section 2 the high part of the spectrum of $(D(A^*), A^*)$, we localize the sufficiently large eigenvalues $(\lambda_n)_{n\geqslant N}$ and we show that the corresponding eigenvectors $(\psi_n)_{n\geqslant N}$ are geometrically simple and quadratically close to the orthonormal sequence $(\phi_n)_{n\geqslant N}=\left(\sqrt{\frac{2}{\pi}}\sin(nx)\right)_{n\geqslant N}$:

$$\sum_{n\geqslant N}\|\psi_n-\phi_n\|^2_{L^2(0,\pi)}<\infty.$$

From [9, Theorem 1] we know that there exist a number $N_H\geqslant N$ and generalized eigenvectors $(\widetilde{\psi}_n)_{1\leqslant n\leqslant N_H-1}$ of the operator $(D(A^*),A^*)$ such that $(\widetilde{\psi}_n)_{1\leqslant n\leqslant N_H-1}\cup(\psi_n)_{n\geqslant N_H}$ forms a Riesz basis $\mathcal{B}$ of $L^2(0,\pi)$. The set of the eigenvalues corresponding to the generalized eigenvectors from $\mathcal{B}$ will be denoted by $\Sigma=(\lambda_n)_{1\leqslant n\leqslant N_L}\cup(\lambda_n)_{n\geqslant N_H}$.

We recall that $(f_n)_{n\geqslant 1}$ is a *Riesz basis* of a Hilbert space H if it is complete in H and there exist two positive constants c_1 and c_2 such that the following inequalities are verified

$$c_1\sum_{n\geqslant 1}|a_n|^2\leqslant\left\|\sum_{n\geqslant 1}a_nf_n\right\|^2_H\leqslant c_2\sum_{n\geqslant 1}|a_n|^2, \tag{1.14}$$

for any finite sequence of scalars $(a_n)_{n\geqslant 1}$. For equivalent definitions and properties of Riesz basis the interested reader is referred to [14, Ch. 1, Sec. 8].

- By expanding the solution φ of (1.10) in the Riesz basis constructed above, we reduce the proof of inequality (1.13) to obtain and evaluate the norm of a biorthogonal sequence $(\theta_{\lambda,j})_{\lambda\in\Sigma,0\leqslant j\leqslant\eta-1}$ to the family of functions $\Lambda=\left(t^je^{-\lambda t}\right)_{\lambda\in\Sigma,\,0\leqslant j\leqslant\eta-1}$ in $L^2\left(-\frac{T}{2},\frac{T}{2}\right)$, where $\eta\geqslant 1$ is the maximal dimension of the root linear space corresponding to the low eigenvalues $(\lambda_n)_{1\leqslant n\leqslant N_L}$. This is done in Theorem 4.8, by adapting some ideas from [1] and [12].

  We recall that, given a sequence $(f_n)_{n\geqslant 1}$ in the Hilbert space H endowed with the inner product $(\,\cdot\,,\,\cdot\,)$, $(g_n)_{n\geqslant 1}$ is a *biorthogonal family* to $(f_n)_{n\geqslant 1}$ in H if the following relations are verified

$$(f_m,g_n)_H=\delta_{mn}\qquad(n,m\geqslant 1). \tag{1.15}$$

From Theorem 1.2, we can obtain the main result of this paper concerning the local controllability of (1.1) to the stationary states:

Theorem 1.3. *Assume that $f^S\in L^2(0,\pi)$ such that (1.2) admits a solution*

$$w^S\in H^2(0,\pi)\cap H^1_0(0,\pi),\quad w^S\neq 0.$$

Let $\omega\subset[0,\pi]$ be a non empty open set such that $\partial_{xx}w^S$ is not identically zero in ω and let $T>0$. Then there exists $c_0>0$ such that for any $w_0\in H^1_0(0,\pi)$ with

$$\|w_0-w^S\|_{H^1_0(0,\pi)}\leqslant c_0,$$

there exists a control $u\in L^2(0,T;L^2(0,\pi))$ such that the solution w of (1.1) satisfies

$$w(T)=w^S.$$

In the above result, we assume that (1.2) admits a solution for some f^S. Note that for an arbitrary $f^S\in L^2(0,\pi)$, (1.2) may not have a solution. Nevertheless, it is easy to construct an infinite number of solutions of (1.2), even with the restriction that $\partial_{xx}w^S$ does not cancel in $(0,\pi)$.

Notice that our controllability result holds for initial data in $H^1_0(0,\pi)$. This is due to the fact that the solution w of (1.1) in Theorem 1.3 should be a strong solution. We obtain such a solution by a fixed point argument and we need in particular that $w_0\in H^1_0(0,\pi)$ and that $w^S\in H^2(0,\pi)\cap H^1_0(0,\pi)$ to handle the nonlinear terms. Let us remark that even the well-posedness for weak solutions of (1.1) (without control) is not an easy issue. Indeed, the compactness or completeness methods can not be applied directly with a regularity such as $w \in L^\infty(0,T;L^2(0,\pi)) \cap L^2(0,T;H_0^1(0,\pi))$ because of the nonlinear term $\nu(\|\partial_x w\|^2_{L^2(0,\pi)})$.

The outline of the paper is the following: in Section 2, we describe the main spectral properties of the linear operator A^* defined by (1.9). In Section 3, we give the spectral decomposition of the solutions of the linearized system (1.10) in terms of a Riesz basis formed from generalized eigenvectors of A^*. Section 4 is devoted to the construction and the evaluation of a biorthogonal family to the set of functions $\Lambda = \left(t^j e^{-\lambda t}\right)_{\lambda\in\Sigma,\, 0\leqslant j\leqslant \eta-1}$. This allows us to prove in Section 5 the observability inequality given by Theorem 1.2. Finally, Section 6 is devoted to the proof of our main result Theorem 1.3.

2 Spectral analysis

The aim of this section is to give a complete description of the high part of the spectrum of the operator $(D(A^*), A^*)$. First of all we have the following quite general result.

Theorem 2.1. *The operator $(D(A^*), A^*)$ defined by (1.7) is an unbounded linear operator in $L^2(0,\pi)$ with compact resolvent. Its spectrum $\sigma(A^*)$ consists of a sequence of isolated complex eigenvalues. To each eigenvalue $\lambda \in \sigma(A^*)$ corresponds a finite dimensional root linear space $\mathcal{G}_\lambda$ (the space of generalized eigenvectors). Moreover, if $D_0 = \|K\|_{L^2((0,\pi)^2)} + \|q\|_{L^\infty(0,\pi)}$, then we have that*

$$\sigma(A^*) = \{\lambda \in \mathbb{C} \ : \ |\Im(\lambda)| \leqslant D_0,\ \Re(\lambda) \geqslant -D_0\}. \tag{2.1}$$

Proof. The fact that $(D(A^*), A^*)$ has compact resolvent follows from the compact embedding of $H^2(0,\pi)$ into $L^2(0,\pi)$. The classical spectral theory of compact operators ensures that the spectrum $\sigma(A^*)$ consists of isolated complex eigenvalues and the root linear space $\mathcal{G}_\lambda$ corresponding to each eigenvalue λ is of finite dimension.

Let us now show that (2.1) holds. Given $\lambda \in \sigma(A^*)$ there exists a function $u \in D(A^*)$ with $\|u\|_{L^2(0,\pi)} = 1$ such that

$$-\partial_{xx}u + \int_0^\pi K(\,\cdot\,,\xi)u(\xi)\,\mathrm{d}\xi + qu = \lambda u. \tag{2.2}$$

Multiplying (2.2) by $\overline{u}$ and integrating by parts we obtain that

$$\int_0^\pi |\partial_x u(x)|^2\,\mathrm{d}x + \int_0^\pi\int_0^\pi K(x,\xi)u(\xi)\overline{u}(x)\,\mathrm{d}\xi\,\mathrm{d}x + \int_0^\pi q(x)|u(x)|^2\,\mathrm{d}x = \lambda. \tag{2.3}$$

By taking the imaginary part of (2.3) we get that

$$\begin{aligned}|\Im(\lambda)| &= \left|\Im\left(\int_0^\pi\int_0^\pi K(x,\xi)u(\xi)\overline{u}(x)\,\mathrm{d}\xi\,\mathrm{d}x + \int_0^\pi q(x)|u(x)|^2\,\mathrm{d}x\right)\right| \\ &\leqslant \|K\|_{L^2((0,\pi)^2)} + \|q\|_{L^\infty(0,\pi)},\end{aligned}$$

and the first relation in (2.1) follows.

By taking the real part of (2.3) we get that

$$\begin{aligned}\Re(\lambda) &= \int_0^\pi |\partial_x u(x)|^2\,\mathrm{d}x + \Re\left(\int_0^\pi\int_0^\pi K(x,\xi)u(\xi)\overline{u}(x)\,\mathrm{d}\xi\,\mathrm{d}x + \int_0^\pi q(x)|u(x)|^2\,\mathrm{d}x\right) \\ &\geqslant -\left(\|K\|_{L^2((0,\pi)^2)} + \|q\|_{L^\infty(0,\pi)}\right),\end{aligned}$$

and the second relation in (2.1) follows. The proof of the theorem is complete. ☐

We pass to localize and describe the high eigenvalues of the operator $(D(A^*), A^*)$. In the sequel, given $a \in \mathbb{C}$ and $r \geqslant 0$, we denote by $B(a,r)$ the ball in the complex plane of center a and radius r. We localize the eigenvalues of the unbounded operator $(D(A^*), A^*)$ by using a strategy similar to the one presented in [11, Chapter 2] which combines the shooting method and the Rouché Theorem. Similar ideas have been employed to analyze the spectrum of several differential operators (see, for instance, [2, 4, 15]). In order to do that let us define the map

$$G : \mathbb{C} \to \mathbb{C}, \; G(\mu) = v(\pi), \tag{2.4}$$

where v is the unique solution of the initial value problem

$$\begin{cases} -\partial_{xx} v(x) - \mu v(x) = 0 & x \in (0,\pi) \\ v(0) = 0 \\ \partial_x v(0) = 1. \end{cases} \tag{2.5}$$

We have the following immediate result.

Proposition 2.2. *The value $\mu \in \mathbb{C}$ is a root of the function G given by (2.4) if and only if it is an eigenvalue of the one dimensional Laplace operator $(D(\widetilde{A}), \widetilde{A})$,*

$$D(\widetilde{A}) = H^2(0,\pi) \cap H_0^1(0,\pi), \quad \widetilde{A}u = -\partial_{xx} u, \tag{2.6}$$

i.e. there exists $n \in \mathbb{N}^*$ such that $\mu = n^2$.

Now, given $D > 0$ and $M > D^2$, let us set

$$\Delta_{M,D} = \{ z \in \mathbb{C} \, : \, \Re(z) \geqslant M, \; |\Im(z)| \leqslant D \}. \tag{2.7}$$

Remark 2.3. *For our future computations we need to give some estimates of $\sqrt{\mu}$ in the case when $\mu \in \Delta_{M,D}$. If $\mu \in \Delta_{M,D}$, it is easy to see that*

$$\Re(\sqrt{\mu}) \geqslant \sqrt{M}, \quad |\Im(\sqrt{\mu})| \leqslant \frac{D}{2\sqrt{M}}. \tag{2.8}$$

Notice that, here and in the sequel, $\sqrt{\mu}$ represents the principal branch of the square root function such that $\sqrt{\mu} \in \mathbb{R}_+$, if $\mu \in \mathbb{R}_+$.

We define the map

$$F : \Delta_{D,M} \to \mathbb{C}, \; F(\mu) = z(\pi), \tag{2.9}$$

where z is the unique solution of the initial value problem

$$\begin{cases} -\partial_{xx} z(x) - \mu z(x) + \displaystyle\int_0^\pi K(x,\xi) z(\xi) \, \mathrm{d}\xi + q(x) z(x) = 0 & x \in (0,\pi) \\ z(0) = 0 \\ \partial_x z(0) = 1. \end{cases} \tag{2.10}$$

Given $D > 0$, let us show that there exists $M > D^2$ such that F is well-defined in $\Delta_{M,D}$, i.e. equation (2.10) has a unique solution $z \in H^2(0,\pi)$ for each $\mu \in \Delta_{M,D}$. We have the following result.

Proposition 2.4. *Let us set*

$$C_{K,q} := 2\pi e^\pi \left(\|K\|_{L^2((0,\pi)^2)} + \|q\|_{L^\infty(0,\pi)} \right) = 2\pi e^\pi D_0. \tag{2.11}$$

Given $D > 0$, $K \in L^2((0,\pi)^2)$ and $q \in L^\infty(0,\pi)$, let $M > 0$ be such that

$$\max\{D, \, C_{K,q}\} < \sqrt{M}. \tag{2.12}$$

Then, for each $\mu \in \Delta_{M,D}$, the integro-differential equation (2.10) *has a unique solution $z \in H^2(0,\pi)$ which verifies the following variation of constants formula*

$$
\begin{aligned}
z(x) &= \frac{1}{\sqrt{\mu}} \sin(\sqrt{\mu} x) \\
&+ \frac{1}{\sqrt{\mu}} \int_0^x \sin(\sqrt{\mu}(x-s)) \left(\int_0^\pi K(s,\xi) z(\xi)\, \mathrm{d}\xi + q(s) z(s) \right) \mathrm{d}s \qquad (x \in (0,\pi)),
\end{aligned} \tag{2.13}
$$

and the following estimate

$$
\|z\|_{L^\infty(0,\pi)} \leqslant \frac{2}{\sqrt{|\mu|}} e^{\frac{\pi}{2}}. \tag{2.14}
$$

Proof. Given $f \in L^2(0,\pi)$ and $\mu \in \Delta_{M,D}$, let us consider the nonhomogeneous equation

$$
\begin{cases}
-\partial_{xx} v(x) - \mu v(x) = f(x) & x \in (0,\pi) \\
v(0) = 0 \\
\partial_x v(0) = 1,
\end{cases} \tag{2.15}
$$

and remark that its unique solution $v \in H^2(0,\pi)$ is given by the following formula

$$
v(x) = \frac{1}{\sqrt{\mu}} \sin(\sqrt{\mu} x) - \frac{1}{\sqrt{\mu}} \int_0^x \sin(\sqrt{\mu}(x-s)) f(s) \mathrm{d}s \qquad (x \in (0,\pi)). \tag{2.16}
$$

With this in mind, we define the map

$$
\begin{gathered}
z \in L^2(0,\pi) \mapsto \mathcal{L} z \in L^2(0,\pi), \\
\mathcal{L} z(x) = \frac{1}{\sqrt{\mu}} \sin(\sqrt{\mu} x) + \frac{1}{\sqrt{\mu}} \int_0^x \sin(\sqrt{\mu}(x-s)) \left(\int_0^\pi K(s,\xi) z(\xi)\, \mathrm{d}\xi + q(s) z(s) \right) \mathrm{d}s.
\end{gathered} \tag{2.17}
$$

We remark that z verifies (2.13) if and only if it is a fixed point of the operator $\mathcal{L}$. We show that, by choosing M as in (2.12), $\mathcal{L}$ becomes a contraction in $L^2(0,\pi)$ for each $\mu \in \Delta_{M,D}$. Indeed, given $z_1, z_2 \in L^2(0,\pi)$ and taking into account (2.8), we obtain that

$$
\begin{aligned}
\|\mathcal{L} z_1 - \mathcal{L} z_2\|_{L^2(0,\pi)} &\leqslant \sqrt{\frac{\pi}{|\mu|}} e^{\pi |\Im(\sqrt{\mu})|} \left(\|K\|_{L^2((0,\pi)^2)} + \|q\|_{L^\infty(0,\pi)} \right) \|z_1 - z_2\|_{L^2(0,\pi)} \\
&\leqslant \sqrt{\frac{\pi}{M}} e^{\frac{\pi D}{2\sqrt{M}}} \left(\|K\|_{L^2((0,\pi)^2)} + \|q\|_{L^\infty(0,\pi)} \right) \|z_1 - z_2\|_{L^2(0,\pi)}.
\end{aligned} \tag{2.18}
$$

Since (2.12) implies that

$$
\sqrt{\frac{\pi}{M}} e^{\frac{\pi D}{2\sqrt{M}}} \left(\|K\|_{L^2((0,\pi)^2)} + \|q\|_{L^\infty(0,\pi)} \right) \leqslant \sqrt{\frac{1}{4\pi}} < 1, \tag{2.19}
$$

from (2.18) it follows that $\mathcal{L}$ is a contraction in $L^2(0,\pi)$ for each $\mu \in \Delta_{M,D}$. Hence, in this case, there exists a unique $z \in L^2(0,\pi)$ which verifies (2.13).

It is easy to see that, if $z \in L^2(0,\pi)$ verifies (2.13), with $K \in L^2((0,\pi)^2)$ and $q \in L^\infty(0,T)$, then in fact $z \in H^2(0,\pi)$ and it is the unique solution of (2.10). Hence, it remains to prove (2.14). We have that, for each $x \in (0,\pi)$, the following estimate holds

$$
|z(x)| \leqslant \frac{1}{\sqrt{|\mu|}} e^{\pi |\Im(\sqrt{\mu})|} + \frac{\pi}{\sqrt{|\mu|}} e^{\pi |\Im(\sqrt{\mu})|} \left(\|K\|_{L^2((0,\pi)^2)} + \|q\|_{L^\infty(0,\pi)} \right) \|z\|_{L^\infty(0,\pi)}. \tag{2.20}
$$

Since (2.12) implies that, for any $\mu \in \Delta_{M,D}$, we have

$$\frac{\pi}{\sqrt{|\mu|}} e^{\pi|\Im(\sqrt{\mu})|} \left(\|K\|_{L^2((0,\pi)^2)} + \|q\|_{L^\infty(0,\pi)}\right) \leqslant \frac{\pi}{\sqrt{M}} e^{\frac{\pi D}{2\sqrt{M}}} \left(\|K\|_{L^2((0,\pi)^2)} + \|q\|_{L^\infty(0,\pi)}\right) \leqslant \frac{1}{2},$$

from (2.20) and (2.8) we deduce that (2.14) holds true. The proof of the proposition is now complete. $\square$

Remark 2.5. *If $|\mu|$ is not sufficiently large, equation* (2.10) *may have multiple solution. For instance, in the case $q \equiv 0$, we have the following result.*

Proposition 2.6. *Assume that $\mu \in \mathbb{C}^*$ and let $\alpha,\ \beta \in L^2(0,\pi)$ be such that*

$$\sqrt{\mu} = \int_0^\pi \beta(x) \int_0^x \sin(\sqrt{\mu}(x-s))\alpha(s)\,\mathrm{d}s\,\mathrm{d}x, \tag{2.21}$$

where $\sqrt{\mu}$ is a square root of μ. Then for any $\vartheta \in \mathbb{C}$, w given by

$$w(x) = \frac{\vartheta}{\sqrt{\mu}} \int_0^x \sin(\sqrt{\mu}(x-s))\alpha(s)\,\mathrm{d}s, \tag{2.22}$$

is a solution of

$$\begin{cases} -\partial_{xx}w(x) + \displaystyle\int_0^\pi K(x,\xi)w(\xi)\,\mathrm{d}\xi = \mu w(x) \quad x \in (0,\pi) \\ w(0) = \partial_x w(0) = 0 \end{cases} \tag{2.23}$$

with $K(x,\xi) = \alpha(x)\beta(\xi)$ (degenerate kernel). In particular, for any $\mu \in \mathbb{C}^*$, there exists $K \in L^2((0,\pi)^2)$ such that (2.23) *has nontrivial solutions.*

Remark 2.7. *If $\mu = 0$, we can replace* (2.21) *by*

$$1 = \int_0^\pi \beta(x) \int_0^x (x-s)\alpha(s)\,\mathrm{d}s\,\mathrm{d}x$$

and (2.22) *by*

$$w(x) = \vartheta \int_0^x (x-s)\alpha(s)\,\mathrm{d}s.$$

Proof of Proposition 2.6. If ϑ denotes the number $\int_0^\pi \beta(\xi)w(\xi)\,\mathrm{d}\xi$, we have that the solution w of (2.23) is given by (2.22). Thus, ϑ is solution of

$$\vartheta = \frac{\vartheta}{\sqrt{\mu}} \int_0^\pi \beta(x) \int_0^x \sin(\sqrt{\mu}(x-s))\alpha(s)\,\mathrm{d}s\,\mathrm{d}x. \tag{2.24}$$

From (2.24) we deduce that there exists two possibilities:

1. Relation (2.21) does not hold. In this case $\vartheta = 0$ and the unique solution of (2.23) is the trivial one.
2. Relation (2.21) holds. In this case any w given by (2.22) with $\vartheta \in \mathbb{C}$ is a solution of (2.23) and the space of solutions of (2.23) is of dimension one.

We notice that, given any $\mu \in \mathbb{C}^*$ and $\alpha \in L^2(0,\pi)$, $\alpha \not\equiv 0$, there exists $\beta \in L^2(0,\pi)$ such that (2.21) holds.

$\square$

Remark 2.8. *We can show that, in the case of a degenerate kernel* $K(x,\xi) = \alpha(x)\beta(\xi)$ *with* $\alpha,\, \beta \in L^2(0,\pi)$*, the geometric multiplicity of any eigenvalue* λ *of the operator* $(D(A^*), A^*)$ *is at most two. Indeed, let* $\lambda \in \mathbb{C}$ *be an eigenvalue of the operator* $(D(A^*), A^*)$*, i.e. there exists* $\varphi \in D(A^*)$ *with* $\|\varphi\|_{L^2(0,\pi)} = 1$ *such that*

$$\left\{ \begin{array}{l} -\partial_{xx}\varphi(x) + \displaystyle\int_0^\pi K(x,\xi)\varphi(\xi)\,\mathrm{d}\xi + q(x)\varphi(x) = \lambda\varphi(x) \quad x \in (0,\pi) \\ \varphi(0) = \varphi(\pi) = 0. \end{array} \right. \tag{2.25}$$

Also, let ϕ^1 *and* ϕ^2 *be the unique solutions of the equations*

$$\left\{ \begin{array}{l} -\partial_{xx}\phi^1(x) + q(x)\phi^1(x) = \lambda\phi^1(x) \quad x \in (0,\pi) \\ \phi^1(0) = 1,\ \partial_x\phi^1(0) = 0, \end{array} \right. \tag{2.26}$$

and

$$\left\{ \begin{array}{l} -\partial_{xx}\phi^2(x) + q(x)\phi^2(x) = \lambda\phi^2(x) \quad x \in (0,\pi) \\ \phi^2(0) = 0,\ \partial_x\phi^2(0) = 1, \end{array} \right. \tag{2.27}$$

respectively. Any solution of the non homogeneous problem

$$\left\{ \begin{array}{l} -\partial_{xx}\phi(x) + q(x)\varphi(x) = \lambda\phi(x) + f(x) \quad x \in (0,\pi) \\ \phi(0) = a,\ \partial_x\phi(0) = b, \end{array} \right. \tag{2.28}$$

can be written as follows

$$\phi(x) = \big(a\phi^1(x) + b\phi^2(x)\big) - \int_0^x H(x,s)f(s)\,\mathrm{d}s \qquad (x \in (0,\pi)), \tag{2.29}$$

where

$$H(x,s) = \frac{\phi^1(s)\phi^2(x) - \phi^1(x)\phi^2(s)}{\phi^1(s)\partial_x\phi^2(s) - \partial_x\phi^1(s)\phi^2(s)}.$$

From this formula, we obtain that any eigenfunction φ *verifying* (2.25) *is of the following form*

$$\varphi(x) = \partial_x\varphi(0)\phi^2(x) + \vartheta\int_0^x H(x,s)\alpha(s)\,\mathrm{d}s \qquad (x \in (0,\pi)), \tag{2.30}$$

where $\vartheta = \int_0^\pi \beta(\xi)\varphi(\xi)\,\mathrm{d}\xi$*.*

By taking into account the above considerations we have that the following two cases are possible for the eigenspace $\mathcal{E}_\lambda = \{v \in D(A^*) \,:\, A^*v = \lambda v\}$ *corresponding to* λ*:*

1. *If* $1 \neq \displaystyle\int_0^\pi \beta(x)\int_0^x H(x,s)\alpha(s)\,\mathrm{d}s\,\mathrm{d}x$*, then the eigenspace* $\mathcal{E}_\lambda$ *is one dimensional and it is generated by the unique solution of the equation*

$$\left\{ \begin{array}{l} -\partial_{xx}w(x) + \displaystyle\int_0^\pi K(x,\xi)w(\xi)\,\mathrm{d}\xi + q(x)w(x) = \lambda w(x) \quad x \in (0,\pi) \\ w(0) = 0,\ \partial_x w(0) = 1 \end{array} \right. \tag{2.31}$$

2. *If* $1 = \displaystyle\int_0^\pi \beta(x)\int_0^x H(x,s)\alpha(s)\,\mathrm{d}s\,\mathrm{d}x$*, then we have the following possibilities*

   (a) *If* $\displaystyle\int_0^\pi \phi^2(s)\beta(s)\,\mathrm{d}s = 0$*,* $\displaystyle\int_0^\pi H(\pi,s)\alpha(s)\,\mathrm{d}s = 0$ *and* $\phi^2(\pi) = 0$ *(i.e.* λ *is an eigenvalue of the operator* $-\partial_{xx} + q$*) then the space* $\mathcal{E}_\lambda$ *is two dimensional and it is generated by the functions*

$$\phi^2(x), \qquad \int_0^x H(x,s)\alpha(s)\,\mathrm{d}s.$$

(b) *Otherwise, the eigenspace $\mathcal{E}_\lambda$ is one dimensional.*

The following immediate property of the function F is very important in the sequel and justifies the introduction of F.

Proposition 2.9. *Assume that* (2.12) *holds true. Then the value $\mu \in \Delta_{M,D}$ is a root of the function F if and only if it is an eigenvalue of the operator $(D(A^*), A^*)$.*

According to Proposition 2.9, if we want to find the eigenvalues of the operator $(D(A^*), A^*)$ from $\Delta_{M,D}$, it is sufficient to look for the roots of F in this domain. We shall localize the large roots of F by using the Rouché Theorem. Firstly, we have to prove the following three lemmas.

Lemma 2.10. *Given $D > 0$ and M satisfying* (2.12), *the following estimate holds*

$$|F(\mu) - G(\mu)| \leqslant \frac{C_{K,q}}{|\mu|} \qquad (\mu \in \Delta_{M,D}). \tag{2.32}$$

Proof. We have that

$$F(\mu) - G(\mu) = z(\pi) - v(\pi) := w(\pi),$$

where w is the unique solution of

$$\begin{cases} -\partial_{xx} w(x) - \mu w(x) + \displaystyle\int_0^\pi K(x,\xi) z(\xi)\,\mathrm{d}\xi + q(x) z(x) = 0 \quad x \in (0,\pi) \\ w(0) = \partial_x w(0) = 0. \end{cases} \tag{2.33}$$

Since the solution of (2.33) is given by

$$w(x) = \frac{1}{\sqrt{\mu}} \int_0^x \sin(\sqrt{\mu}(x-s)) \left(\int_0^\pi K(s,\xi) z(\xi)\,\mathrm{d}\xi + q(s) z(s) \right) \mathrm{d}s \qquad (x \in (0,\pi)),$$

it follows that

$$|w(x)| \leqslant \frac{\pi}{\sqrt{|\mu|}} e^{\pi |\Im(\sqrt{\mu})|} \left(\|K\|_{L^2((0,\pi)^2)} + \|q\|_{L^\infty(0,\pi)} \right) \|z\|_{L^\infty(0,\pi)} \qquad (x \in (0,\pi)). \tag{2.34}$$

From (2.14) and (2.34) it follows that

$$|w(\pi)| \leqslant \frac{2\pi}{|\mu|} e^{\frac{\pi D}{2\sqrt{M}} + \frac{\pi}{2}} \left(\|K\|_{L^2((0,\pi)^2)} + \|q\|_{L^\infty(0,\pi)} \right).$$

Since $M > D^2$, the above inequality implies that (2.32) holds. □

Given $\delta > 0$, for each $n \in \mathbb{N}$, we define the circles

$$\Gamma_n(\delta) = \left\{ z \in \mathbb{C} \,:\, |z - n^2| = \delta \right\} = \partial B(n^2, \delta). \tag{2.35}$$

Also, we define the set

$$S(\delta) = \left\{ z \in \Delta_{M,D} \,:\, |z - n^2| \geqslant \delta, \, \forall n \geqslant 0 \right\}. \tag{2.36}$$

Lemma 2.11. *Given $\delta > 0$, $D > 0$ and M satisfying* (2.12), *the function G defined by* (2.4) *verifies the following inequality*

$$|G(\mu)| \geqslant \frac{2\delta}{3|\mu|} \qquad \left(\mu \in S(\delta), \quad |\mu| \geqslant \frac{1}{2} \right). \tag{2.37}$$

Proof. We have that $G(\mu) = v(\pi) = \frac{1}{\sqrt{\mu}} \sin(\pi\sqrt{\mu})$ and the problem is reduced to evaluate $\sin(\pi\sqrt{\mu})$ in $S(\delta)$. Let $\mu \in S(\delta)$ and let us set that $\sqrt{\mu} = a + ib$ with $b \in \mathbb{R}$ and $a \in \mathbb{R}_+$. If $n_a \in \mathbb{N}$ is a number which verifies

$$|a - n_a| \leqslant \frac{1}{2}, \tag{2.38}$$

we have that

$$\begin{aligned} |\sin(\sqrt{\mu}\pi)|^2 = |\sin((a+ib)\pi)|^2 &= \frac{1}{4}\left(e^{2b\pi} + e^{-2b\pi} - 2\cos(2a\pi)\right) \\ &\geqslant b^2\pi^2 + \sin^2(a\pi) = b^2\pi^2 + \sin^2((a-n_a)\pi). \end{aligned} \tag{2.39}$$

Inequality (2.38), together with the facts that $\mu \in \Delta_{M,D}$ (see (2.8)) and $M > D^2$, implies that

$$|\sqrt{\mu} - n_a|^2 = |a - n_a|^2 + |b|^2 \leqslant \frac{1}{4} + \left(\frac{D}{2\sqrt{M}}\right)^2 \leqslant \frac{1}{2}.$$

Since, for $|\sqrt{\mu}| \geqslant \frac{1}{\sqrt{2}}$, we have that

$$|\sqrt{\mu} + n_a| \leqslant |\sqrt{\mu} - n_a| + 2|\sqrt{\mu}| \leqslant \frac{1}{\sqrt{2}} + 2|\sqrt{\mu}| \leqslant 3|\sqrt{\mu}|,$$

from (2.39) and (2.38) we deduce that, for any $\mu \in S(\delta)$, we have that

$$|\sin(\sqrt{\mu}\pi)|^2 \geqslant 4b^2 + 4(a-n_a)^2 = 4\left|\sqrt{\mu} - n_a\right|^2 \geqslant \frac{4\delta^2}{\left|\sqrt{\mu} + n_a\right|^2} \geqslant \frac{4\delta^2}{9\,|\mu|}.$$

From the last inequality it follows immediately that (2.37) is verified if $|\sqrt{\mu}| \geqslant \frac{1}{\sqrt{2}}$ and the proof ends. □

Lemma 2.12. *Given $D > 0$ and M satisfying* (2.12)*, the functions $G : \mathbb{C} \to \mathbb{C}$ and $F : \Delta_{M,D} \to \mathbb{C}$, given by* (2.4) *and* (2.9) *respectively, are analytic in their domains of definition.*

Proof. Since $G(\mu) = \frac{1}{\sqrt{\mu}} \sin(\sqrt{\mu}\pi)$, it follows immediately that G is an entire function. Let us prove the property for F. We recall that, according to Proposition 2.4, for each $\mu \in \Delta_{M,D}$, there exists a unique solution $z = z(\,\cdot\,, \mu) \in H^2(0,\pi)$ of equation (2.13). Let us define the Picard's recurrent sequence of functions

$$\left\{ \begin{aligned} & z_0(x,\mu) = \frac{1}{\sqrt{\mu}} \sin(\sqrt{\mu}x), \\ & z_{n+1}(x,\mu) = z_0(x,\mu) \\ & \qquad + \int_0^x \frac{\sin(\sqrt{\mu}(x-s))}{\sqrt{\mu}} \left(\int_0^\pi K(s,\xi) z_n(\xi,\mu)\,\mathrm{d}\xi + q(s) z_n(s,\mu)\right) \mathrm{d}s, \qquad (n \geqslant 0). \end{aligned} \right. \tag{2.40}$$

We have that

$$\begin{aligned} &\|(z_{n+1} - z_n)(\,\cdot\,,\mu)\|_{L^\infty(0,\pi)} \\ &\qquad \leqslant \frac{\pi}{\sqrt{|\mu|}} e^{|\Im(\sqrt{\mu})|\pi} \left(\|K\|_{L^2((0,\pi)^2)} + \|q\|_{L^\infty(0,\pi)}\right) \|(z_n - z_{n-1})(\,\cdot\,,\mu)\|_{L^\infty(0,\pi)}. \end{aligned} \tag{2.41}$$

By taking into account (2.8) and (2.12), it follows from (2.41) that

$$\|(z_{n+1} - z_n)(\,\cdot\,,\mu)\|_{L^\infty(0,\pi)} \leqslant \frac{1}{2} \|(z_n - z_{n-1})(\,\cdot\,,\mu)\|_{L^\infty(0,\pi)}. \tag{2.42}$$

Hence, the series $\sum_{n\geqslant 0}(z_{n+1}-z_n)(\,\cdot\,,\mu)$ converges in $\mathcal{C}[0,\pi]$ and since

$$\|z_0(\,\cdot\,,\mu)\|_{L^\infty(0,\pi)} \leqslant \frac{1}{\sqrt{M}}e^{\pi|\Im\sqrt{\mu}|} \leqslant \frac{1}{\sqrt{M}}e^{\frac{\pi}{2}},$$

this convergence is uniformly in $\mu \in \Delta_{M,D}$. Since

$$\sum_{n=0}^{k}(z_{n+1}-z_n)(\,\cdot\,,\mu) = z_{k+1}(\,\cdot\,,\mu) - z_0(\,\cdot\,,\mu),$$

it follows that the sequence $(z_n(\,\cdot\,,\mu))_{n\geqslant 0}$ converges in $\mathcal{C}[0,\pi]$ as n tends to infinity, uniformly in $\mu \in \Delta_{M,D}$. In fact, the limit of the sequence $(z_n(\,\cdot\,,\mu))_{n\geqslant 0}$ is the unique solution $z(\,\cdot\,,\mu)$ of (2.13) encountered in Proposition 2.4. It follows that the sequence $(z_n(\pi,\mu))_{n\geqslant 0}$ converges to $z(\pi,\mu)$ as n tends to infinity, uniformly in $\mu \in \Delta_{M,D}$. Since each z_n depends analytically of μ in $\Delta_{M,D}$, it follows that $F(\mu) = z(\pi,\mu)$ is analytic in $\Delta_{M,D}$ and the proof of the lemma is now complete. □

We have now all the ingredients needed to apply the Rouché Theorem and to localize the zeros of F from the region $\Delta_{M,D}$. In the sequel $\lfloor\cdot\rfloor$ denotes the floor function.

Theorem 2.13. *There exists $\delta > 0$, such that for any $D > \delta$ and $N \geqslant \max\{\lfloor 2\delta\rfloor, \lfloor D\rfloor + 1\}$, there exists M satisfying* (2.12) *and having the following property: for each $n \geqslant N+1$, the function F has a unique root λ_n inside the circle $\Gamma_n(\delta)$ defined by* (2.35)*. Moreover, the roots of the function F in $\Delta_{M,D}$ are exactly $(\lambda_n)_{n\geqslant N+1}$.*

Proof. We recall that $C_{K,q}$ is defined by (2.11). We choose $\delta = \frac{3C_{K,q}}{2} + 1$ and let $D > \delta$. From the choice of N we deduce that $(N+1)^2 - \delta > N^2 + \delta$ and we can take M such that

$$(N+1)^2 - \delta > M > N^2 + \delta.$$

From the above relations we have, in particular, that M satisfies (2.12) and

$$\Gamma_n(\delta) \subset \Delta_{M,D} \qquad (n \geqslant N+1).$$

From Lemmas 2.10, 2.11 and the choice of δ it follows that, for each $n \geqslant N+1$, we have that

$$|F(\mu) - G(\mu)| \leqslant \frac{C_{K,q}}{|\mu|} < \frac{2\delta}{3|\mu|} \leqslant |G(\mu)| \qquad (\mu \in \Gamma_n(\delta)). \tag{2.43}$$

Moreover, Lemma 2.12 ensures that F and G are analytic functions in $\Delta_{M,D}$. From the Rouché Theorem we deduce that, for each $n \geqslant N+1$, there exists a simple root λ_n of F inside the contour $\Gamma_n(\delta) \subset \Delta_{M,D}$. On the other hand, by using estimate (2.37) from Lemma 2.11 it follows that

$$|F(\mu) - G(\mu)| < |G(\mu)| \qquad (\mu \in S(\delta)). \tag{2.44}$$

This implies that there are no other roots of the function F in $\Delta_{M,D}$. □

We pass to prove some properties of the associated eigenvectors.

Lemma 2.14. *Under the hypothesis of Theorem 2.13, for each $n \geqslant N+1$, let $\phi_n(x) = \sqrt{\frac{2}{\pi}}\sin(nx)$ and $\psi_n(x) = \sqrt{\frac{2}{\pi}}\,nz(x,\lambda_n)$, where $z(\,\cdot\,,\lambda_n)$ is the solution of* (2.10) *with $\mu = \lambda_n$. The following properties are verified*

1. *There exist two positive constants ρ_1 and ρ_2 such that*

$$\rho_1 \leqslant \|\psi_n\|_{L^2(0,\pi)} \leqslant \rho_2 \qquad (n \geqslant N+1); \tag{2.45}$$

2. There exists a positive constant ρ_3 such that

$$\|\psi_n - \phi_n\|_{L^2(0,\pi)} \leqslant \frac{\rho_3}{n} \qquad (n \geqslant N+1). \tag{2.46}$$

3. If ω is a non empty open subset of $[0,\pi]$, then there exists a positive constant ρ_4 such that

$$\int_\omega |\psi_n(x)|^2 \,\mathrm{d}x \geqslant \rho_4 \qquad (n \geqslant N+1). \tag{2.47}$$

Proof. We remark that ϕ_n is the solution of the equation

$$\begin{cases} -\partial_{xx}\phi_n(x) - n^2\phi_n(x) = 0 \quad x \in (0,\pi) \\ \phi_n(0) = \phi_n(\pi) = 0 \\ \partial_x\phi_n(0) = \sqrt{\frac{2}{\pi}}n, \end{cases} \tag{2.48}$$

whereas ψ_n is the solution of the equation

$$\begin{cases} -\partial_{xx}\psi_n(x) - \lambda_n\psi_n(x) + \displaystyle\int_0^\pi K(x,\xi)\psi_n(\xi)\,\mathrm{d}\xi + q(x)\psi_n(x) = 0 \quad x \in (0,\pi) \\ \psi_n(0) = \psi_n(\pi) = 0 \\ \partial_x\psi_n(0) = \sqrt{\frac{2}{\pi}}n. \end{cases} \tag{2.49}$$

To prove (2.45), we use (2.49) to deduce that

$$\begin{aligned} \psi_n(x) &= \phi_n(x) \\ &+ \frac{1}{n}\int_0^x \sin(n(x-s))\left((n^2-\lambda_n)\psi_n(s) + \int_0^\pi K(s,\xi)\psi_n(\xi)\,\mathrm{d}\xi + q(s)\psi_n(s)\right)\mathrm{d}s. \end{aligned} \tag{2.50}$$

From (2.50) and the fact that $|n^2-\lambda_n| \leqslant \delta$ it follows that

$$\left|\|\psi_n\|_{L^2(0,\pi)} - \|\phi_n\|_{L^2(0,\pi)}\right| \leqslant \frac{\pi}{n}\left(\delta + \|K\|_{L^2((0,\pi)^2)} + \|q\|_{L^\infty(0,\pi)}\right)\|\psi_n\|_{L^2(0,\pi)}.$$

From the last estimate, by using that $\|\phi_n\|_{L^2(0,\pi)} = 1$ and that, for $n \geqslant \max\{N+1, 4\pi\delta\}$,

$$\pi\left(\delta + \|K\|_{L^2((0,\pi)^2)} + \|q\|_{L^\infty(0,\pi)}\right) < 2\pi\delta \leqslant \frac{n}{2},$$

we obtain that (2.45) holds with

$$\rho_1 = \min\left\{\frac{2}{3}, \min_{N+1\leqslant n\leqslant 4\pi\delta}\|\psi_n\|_{L^2(0,\pi)}\right\}, \quad \rho_2 = \max\left\{2, \max_{N+1\leqslant n\leqslant 4\pi\delta}\|\psi_n\|_{L^2(0,\pi)}\right\}.$$

Let us pass to prove (2.46). If we put $\zeta_n = \psi_n - \phi_n$, it follows that ζ_n verifies the following relation

$$\zeta_n(x) = \frac{1}{n}\int_0^x \sin(n(x-s))\left(\int_0^\pi K(s,\xi)\psi_n(\xi)\,\mathrm{d}\xi + q(s)\psi_n(s) + (n^2-\lambda_n)\psi_n(s)\right)\mathrm{d}s. \tag{2.51}$$

Since, according to Theorem 2.13, $|\lambda_n - n^2| \leqslant \delta$, we deduce from (2.51) that

$$\|\zeta_n\|_{L^2(0,\pi)} \leqslant \frac{\pi}{n}\left(\|K\|_{L^2((0,\pi)^2)} + \|q\|_{L^\infty(0,\pi)} + \delta\right)\|\psi_n\|_{L^2(0,\pi)}. \tag{2.52}$$

From (2.52) we deduce, by taking into account the second inequality in (2.45), that (2.46) holds true with $\rho_3 = 2\pi\delta\rho_2$.

On the other hand, from (2.46), we deduce that

$$\int_\omega |\psi_n(x)|^2 \,\mathrm{d}x \geqslant \frac{1}{\pi}\int_\omega |\sin(nx)|^2 \,\mathrm{d}x - \int_\omega \left|\psi_n(x) - \sqrt{\frac{2}{\pi}}\sin(nx)\right|^2 \mathrm{d}x \geqslant \frac{1}{\pi}\int_\omega |\sin(nx)|^2 \,\mathrm{d}x - \frac{\rho_3^2}{n^2},$$

from which we deduce that there exists $N_1 \geqslant N+1$ and a constant $\rho' > 0$ such that

$$\int_\omega |\psi_n(x)|^2 \,\mathrm{d}x \geqslant \rho' \qquad (n \geqslant N_1). \tag{2.53}$$

Relation (2.53) follows with

$$\rho_3 = \min\left\{\rho',\ \min_{N+1\leqslant n\leqslant N_1}\left\{\int_\omega |\psi_n(x)|^2 \,\mathrm{d}x\right\}\right\}$$

and the proof of the lemma is complete. □

To solve our control problem we shall also need the following unique continuation principle for the solutions of equation (2.10).

Lemma 2.15. *Suppose that the kernel K is degenerate, i.e. there exist two functions $\alpha, \beta \in L^2(0,\pi)$ such that $K(x,\xi) = \alpha(x)\beta(\xi)$ for each $(x,\xi) \in [0,\pi]^2$. Let $\mu \in \mathbb{C}$ and let $z \in H^2(0,\pi)$ be any solution of equation*

$$-\partial_{xx} z(x) + \int_0^\pi K(x,\xi) z(\xi)\,\mathrm{d}\xi + q(x)z(x) = \mu z(x) \qquad (x \in (0,\pi)), \tag{2.54}$$

with the property that there exists a non empty open interval $(a,b) \subset [0,\pi]$ such that z vanishes in (a,b) and α is not identically zero in (a,b). Then z is identically zero in $[0,\pi]$.

Proof. By taking into account the particular form of the kernel K and equation (2.54) verified by z, we obtain that

$$\alpha(x)\int_0^\pi \beta(\xi)z(\xi)\,\mathrm{d}\xi = 0 \qquad (x \in (a,b)).$$

Since α is not identically zero in (a,b) it follows that $\int_0^\pi \beta(\xi)z(\xi)\,\mathrm{d}\xi = 0$. Let $x_0 \in (a,b)$ and notice that $z(x_0) = \partial_x z(x_0) = 0$. From the uniqueness of solutions of the equation

$$\begin{cases} -\partial_{xx} z(x) + (q(x)-\mu)z(x) = 0 \\ z(x_0) = \partial_x z(x_0) = 0, \end{cases}$$

in $(0,x_0)$ and (x_0,π), we deduce that z is identically zero in $[0,\pi]$ and the proof ends. □

Remark 2.16. *The degeneracy hypothesis of the kernel K is verified in the case of our model* (1.6)*. It seems that, for kernels that are not degenerate, additional conditions should be imposed in order to ensure that the unique continuation property holds. For instance, in [8] an assumption of analyticity of the kernel is used in order to solve a multidimensional control problem.*

Remark 2.17. *The hypothesis that α is not identically zero in (a,b) is necessary since it is easy to construct in this case a nonzero function z verifying* (2.54) *which vanishes in (a,b).*

Let us summarize the most important results concerning the high part of the spectrum of the operator $(D(A^*), A^*)$ in the following theorem.

Theorem 2.18. *There exist $N \in \mathbb{N}$ and $\delta > 0$ such that, for each $n \geqslant N+1$, $\sigma(A^*) \cap B(n^2, \delta)$ is reduced to one element that we denote by λ_n and*

$$\sigma(A^*) \setminus B(0, (N+1)^2 - \delta) = \{\lambda_n, \ n \geqslant N+1\}. \tag{2.55}$$

Moreover, for any $n \geqslant N+1$, the eigenvalue λ_n is geometrically simple and there exists an eigenvector ψ_n of A^* corresponding to λ_n such that

$$\|\psi_n - \phi_n\|_{L^2(0,\pi)} \leqslant \frac{\rho}{n}, \tag{2.56}$$

where $\phi_n(x) = \sqrt{\frac{2}{\pi}} \sin(nx)$ and ρ is a positive constant independent of n.

Proof. Let $D_0 > 0$ be the constant from Theorem 2.1. Also, let D, M, $N \in \mathbb{N}$ and $\delta > 0$ be given by Theorem 2.13 with $D \geqslant D_0$. According to Proposition 2.9 and Theorem 2.13 there exists a a unique eigenvalue λ_n of the operator $(D(A^*), A^*)$ in each ball $B(n^2, \delta)$ for $n \geqslant N+1$ and there are no others eigenvalues of this operator in $\Delta_{M,D}$. By taking into account relations (2.1) from Theorem 2.1, we obtain that (2.55) is verified, too. Due to the uniqueness of solutions to problem (2.10) proved in Proposition 2.4, if $n \geqslant N+1$, to each λ_n corresponds a one dimensional eigenspace generated by the function ψ_n defined in Lemma 2.14. Finally, estimate (2.56) for ψ_n follows from (2.46) and the proof of the theorem is complete. $\square$

3 Riesz basis

The analysis of the eigenfunctions $(\psi_n)_{n \geqslant N+1}$ in the previous section and a result from [9, Theorem 1] allow us to obtain a Riesz basis of $L^2(0, \pi)$.

Theorem 3.1. *Let N be the entire number and $(\psi_n)_{n \geqslant N+1}$ be the eigenvectors given by Theorem 2.18. There exist N_H, $N_L \in \mathbb{N}$, with $N_H > \max\{N+1, N_L\}$ and a Riesz basis $\mathcal{B}$ of $L^2(0,\pi)$ such that*

$$\mathcal{B} = (\psi_k)_{k \geqslant N_H} \cup (\widetilde{\psi}_k)_{1 \leqslant k \leqslant N_H - 1}, \tag{3.1}$$

where $(\widetilde{\psi}_k)_{1 \leqslant k \leqslant N_H - 1}$ are generalized eigenvectors of the operator $(D(A^*), A^*)$ corresponding to N_L different eigenvalues of A^*, $\lambda_1, \ldots, \lambda_{N_L}$, satisfying

$$|\lambda_n| < |\lambda_{N_H}| \qquad (1 \leqslant n \leqslant N_L). \tag{3.2}$$

Proof. The result follows immediately from [9, Theorem 1], by taking into account that $(D(A^*), A^*)$ is a densely defined operator with compact resolvent in $L^2(0,\pi)$, $(\phi_n)_{n \geqslant 1}$ is an orthonormal basis of $L^2(0,\pi)$ and $(\psi_k)_{k \geqslant N+1}$ verify (2.56). $\square$

Let us use Theorem 3.1 in order to expand the functions from $L^2(0,\pi)$ and to write the solutions of (1.10). Firstly, let us denote the family of distinct eigenvalues corresponding to all the eigenvectors $(\psi_k)_{k \geqslant N_H}$ and generalized eigenvectors $(\widetilde{\psi}_k)_{1 \leqslant k \leqslant N_H - 1}$ from Theorem 3.1 by

$$\Sigma = \Sigma^L \cup \Sigma^H = (\lambda_k)_{1 \leqslant k \leqslant N_L} \cup (\lambda_k)_{k \geqslant N_H} \subset \mathbb{C}. \tag{3.3}$$

We recall that, according to Theorem 2.18, if $k \geqslant N_H$, the element ψ_k from the basis $\mathcal{B}$ generates the one dimensional eigenspace corresponding to λ_k.

On the other hand, for any $1 \leqslant k \leqslant N_L$, let us consider a Jordan basis of the root space of A^* associated to the eigenvalue λ_k:

$$\psi_{k,j,m} \quad 1 \leqslant k \leqslant N_L, \ 1 \leqslant j \leqslant d_k, \ 0 \leqslant m \leqslant r_{k,j} - 1. \tag{3.4}$$

More precisely, for a given k, $\{\psi_{k,j,m}\}$ is a family of generalized eigenvectors of A^* associated with the eigenvalue λ_k with the following property

$$\psi_{k,j,0} \in \ker(A^* - \lambda_k \mathcal{I}), \quad (A^* - \lambda_k \mathcal{I})\psi_{k,j,m} = -\psi_{k,j,m-1} \quad 1 \leqslant m \leqslant r_{k,j} - 1,\ 1 \leqslant j \leqslant d_k,$$

and the family in (3.4) is linearly independent. Note that

$$\ker(A^* - \lambda_k \mathcal{I}) = \mathrm{Span}\{\psi_{k,j,0}, 1 \leqslant j \leqslant d_k\}.$$

We remark that not all the generalized eigenvectors $(\psi_{k,j,m})_{1 \leqslant k \leqslant N_L,\ 1 \leqslant j \leqslant d_k,\ 0 \leqslant m \leqslant r_{k,j}-1}$ belong necessarily to the Riesz basis $\mathcal{B}$ given by Theorem 3.1. However, in order to simplify the notation, we shall write that any function $\varphi_0 \in L^2(0,\pi)$ can be uniquely written as

$$\varphi_0 = \sum_{k=1}^{N_L} \sum_{j=1}^{d_k} \sum_{m=0}^{r_{k,j}-1} a_{k,j,m} \psi_{k,j,m} + \sum_{k=N_H}^{\infty} a_k \psi_k, \tag{3.5}$$

where in (3.5) we shall consider that $a_{k,j,m} = 0$ if $\psi_{k,j,m} \notin \mathcal{B}$.

Now, if $\varphi_0 \in L^2(0,\pi)$ is given by (3.5), then the corresponding solution φ of (1.10) can be written under the following form

$$\varphi(t) = \sum_{k=1}^{N_L} \sum_{j=1}^{d_k} \sum_{m=0}^{r_{k,j}-1} a_{k,j,m} \left(\sum_{s=0}^{m} \frac{t^s}{s!} \psi_{k,j,m-s} \right) e^{-\lambda_k t} + \sum_{k=N_H}^{\infty} a_k \psi_k e^{-\lambda_k t}. \tag{3.6}$$

We shall need to rewrite the solution φ in a slightly different way. Let us set for $k = 1, \dots, N_L$,

$$R_k := \max\{r_{k,j},\ j = 1, \dots, d_k\}, \tag{3.7}$$

and

$$\mathcal{I}_{k,p} := \{j \in \{1, \dots, d_k\}\ ;\ r_{k,j} = R_k - p\} \quad (0 \leqslant p \leqslant R_k - 1). \tag{3.8}$$

Then we can rewrite (3.6) as

$$\varphi(t) = \sum_{k=1}^{N_L} \sum_{s=0}^{R_k-1} t^s e^{-\lambda_k t} \left(\sum_{p=0}^{R_k-s-1} \sum_{m=s}^{R_k-p-1} \sum_{j \in \mathcal{I}_{k,p}} \frac{1}{s!} a_{k,j,m} \psi_{k,j,m-s} \right) + \sum_{k=N_H}^{\infty} a_k \psi_k e^{-\lambda_k t}. \tag{3.9}$$

For any $k \in \{1, \dots, N_L\}$, the coefficient corresponding to $t^{R_k-1} e^{-\lambda_k t}$ is

$$\sum_{j \in \mathcal{I}_{k,0}} \frac{1}{(R_k-1)!} a_{k,j,R_k-1} \psi_{k,j,0},$$

the coefficient corresponding to $t^{R_k-2} e^{-\lambda_k t}$ is

$$\sum_{j \in \mathcal{I}_{k,0} \cup \mathcal{I}_{k,1}} \frac{1}{(R_k-2)!} a_{k,j,R_k-2} \psi_{k,j,0} + \sum_{j \in \mathcal{I}_{k,0}} \frac{1}{(R_k-2)!} a_{k,j,R_k-1} \psi_{k,j,1},$$

and so on.

We shall use these expressions of the solution φ of (1.10) in order to prove our observability inequality for this equation. Note that in (3.9) appears the family of functions

$$\left(t^s e^{-\lambda_k t}\right)_{1 \leqslant k \leqslant N_L,\ 0 \leqslant s \leqslant R_k - 1} \cup \left(e^{-\lambda_k t}\right)_{k \geqslant N_H}.$$

Given $T > 0$, we need to construct and estimate a biorthogonal family in $L^2(0,T)$ to this family. In fact, we can obtain a slightly more general result and we start with the family

$$\Lambda = \left(t^s e^{-\lambda t}\right)_{\lambda \in \Sigma,\, 0 \leqslant s \leqslant \eta - 1},$$

where $\eta \geqslant 1$ is a given integer. The following section is devoted to the construction and evaluation of a biorthogonal sequence to Λ in $L^2(0,T)$. Let us remark that such a construction has been already done in similar contexts by [1]. However, since we need to estimate the dependence of the norm of the biorthogonal sequence with respect to small T, we shall adopt a different and more explicit approach which is closer to the one used by [12], even though in [12] the exponents λ are purely real and $\eta = 1$.

4 The biorthogonal family

Let Σ be defined by (3.3) and $\eta \in \mathbb{N}^*$. Before starting the construction of a biorthogonal sequence to the family $\Lambda = \left(t^s e^{-\lambda t}\right)_{\lambda \in \Sigma,\, 0 \leqslant s \leqslant \eta - 1}$, for reader's convenience, let us enumerate some simple properties of the set Σ which will be used in this section.

Lemma 4.1. *The eigenvalues of the operator $(D(A^*), A^*)$ from Σ verify the following properties:*

P1) There exists $\gamma > 0$ such that

$$|\lambda - \lambda'| \geqslant \gamma \qquad (\lambda, \lambda' \in \Sigma, \quad \lambda \neq \lambda'). \tag{4.1}$$

P2) There exists a constant $\delta \in (0, N_H)$ such that

$$|\lambda_n - n^2| \leqslant \delta \qquad (n \geqslant N_H), \tag{4.2}$$

$$|\Re(\lambda_{N_H}) - \Re(\lambda_n)| \geqslant \delta \qquad (1 \leqslant n \leqslant N_L). \tag{4.3}$$

P3) For each $1 \leqslant n \leqslant N_L$ we have that

$$|\lambda_n| < |\lambda_{N_H}|. \tag{4.4}$$

P4) The sequence $(\lambda_n)_{n \geqslant N_H}$ verifies the following properties

(a) $|\Re(\lambda_n) - n^2| \leqslant C$, $n \geqslant N_H$,

(b) $\Re(\lambda_{n+1}) - \Re(\lambda_n) \geqslant \delta$, $n \geqslant N_H$.

P5) We have that

$$\sum_{\lambda \in \Sigma^H} \frac{1}{|\lambda|} < \infty. \tag{4.5}$$

Proof. The existence of $\gamma > 0$ verifying (4.1) follows from the localization of the elements of Σ^H given by Theorem 2.18 and the fact that Σ^L has only a finite number of different eigenvalues. On the other hand, with $\delta > 0$ given by Theorem 2.13, both (4.2) and (4.3) are verified. Indeed, since $2\delta < N + 1 < N_H$, we deduce that

$$\operatorname{dist}\left(\Gamma_{N_H}(\delta), \Gamma_n(\delta)\right) \geqslant \delta \qquad (1 \leqslant n \leqslant N_L),$$

which gives (4.3). Property (4.4) follows from Theorem 3.1 and the last properties are direct consequences of Theorem 2.18. $\square$

For each $\lambda \in \Sigma$, let us consider the function $\Phi_\lambda : \mathbb{C} \to \mathbb{C}$ defined by

$$\Phi_\lambda(z) = \prod_{\lambda' \in \Sigma \setminus \{\lambda\}} \left(1 - \frac{z}{\overline{\lambda'} - \overline{\lambda}}\right). \tag{4.6}$$

Some of the basic properties of Φ_λ are described in the following lemma.

Lemma 4.2. *The function Φ_λ is an entire function of arbitrarily small exponential type. Moreover, there exist three positive constants B_0, C_1 and C_2, depending at most upon γ, N_L, N_H and δ, such that we have, uniformly with respect to λ,*

$$|\Phi_\lambda(z)| \leqslant C_1 e^{\pi\sqrt{|z|}}(1+|z|)^{B_0} \qquad (z\in\mathbb{C}), \tag{4.7}$$

$$\left|\Phi_\lambda(ix-\overline{\lambda})\right| \leqslant C_2 e^{\pi\sqrt{|x|}}(1+|\Re(\lambda)|+|x|)^{B_0} \qquad (x\in\mathbb{R}). \tag{4.8}$$

Proof. Firstly, by using (4.1), we deduce that

$$\begin{aligned}|\Phi_\lambda(z)| &\leqslant \prod_{\lambda'\in\Sigma\setminus\{\lambda\}}\left(1+\frac{|z|}{|\overline{\lambda'}-\overline{\lambda}|}\right) \leqslant \prod_{\lambda'\in\Sigma^L\setminus\{\lambda\}}\left(1+\frac{|z|}{\gamma}\right)\prod_{\lambda'\in\Sigma^H\setminus\{\lambda\}}\left(1+\frac{|z|}{|\lambda'-\lambda|}\right)\\ &\leqslant \left(\max\left\{1,\frac{1}{\gamma}\right\}\right)^{N_L}(1+|z|)^{N_L}\prod_{\lambda'\in\Sigma^H\setminus\{\lambda\}}\left(1+\frac{|z|}{|\Re(\lambda')-\Re(\lambda)|}\right).\end{aligned}$$

Now, by taking into account that $(\Re(\lambda))_{\lambda\in\Sigma^H}$ verifies the properties $P4)$ from Lemma 4.1, estimate (4.7) follows as in [12, Lemma 4.1].

For the proof of (4.8), we remark that

$$\begin{aligned}\left|\Phi_\lambda(ix-\overline{\lambda})\right| &= \prod_{\lambda'\in\Sigma\setminus\{\lambda\}}\left|\frac{\overline{\lambda'}-ix}{\overline{\lambda'}-\overline{\lambda}}\right| \leqslant \prod_{\lambda'\in\Sigma^L\setminus\{\lambda\}}\frac{|\lambda'+ix|}{\gamma}\prod_{\lambda'\in\Sigma^H\setminus\{\lambda\}}\left|\frac{\lambda'+ix}{\lambda'-\lambda}\right|\\ &\leqslant \left(1+\frac{1}{\gamma}\right)^{N_L}\left(\max\{1,|\lambda_{N_H}|\}\right)^{N_L}(1+|x|)^{N_L}\prod_{\lambda'\in\Sigma^H}\left(1+\frac{|x|}{|\lambda'|}\right)\prod_{\lambda'\in\Sigma^H\setminus\{\lambda\}}\frac{|\lambda'|}{|\lambda'-\lambda|}.\end{aligned} \tag{4.9}$$

Let us evaluate each of the last two products in (4.9). For the first product we remark that

$$\ln\left[\prod_{\lambda'\in\Sigma^H}\left(1+\frac{|x|}{|\lambda'|}\right)\right] = \underbrace{\sum_{k\geqslant N_H}\ln\left(1+\frac{|x|}{k^2}\right)}_{S_1} + \underbrace{\sum_{k\geqslant N_H}\ln\left(\frac{|\lambda_k|+|x|}{k^2+|x|}\right)}_{S_2} + \underbrace{\sum_{k\geqslant N_H}\ln\left(\frac{k^2}{|\lambda_k|}\right)}_{S_3}. \tag{4.10}$$

Now, from (4.2), we have that there exists a positive constant C such that

$$S_3 \leqslant \sum_{k\geqslant N_H}\ln\left(\frac{|\lambda_k|+|k^2-\lambda_k|}{|\lambda_k|}\right) \leqslant \sum_{k\geqslant N_H}\ln\left(1+\frac{\delta}{|\lambda_k|}\right) \leqslant \sum_{k\geqslant N_H}\frac{\delta}{|\lambda_k|} \leqslant C,$$

$$S_2 = \sum_{k\geqslant N_H}\ln\left(1+\frac{|\lambda_k|-k^2}{k^2+|x|}\right) \leqslant \sum_{k\geqslant N_H}\ln\left(1+\frac{\delta}{k^2+|x|}\right) \leqslant \sum_{k\geqslant N_H}\frac{\delta}{k^2} \leqslant C,$$

$$S_1 \leqslant \sum_{k\geqslant 1}\ln\left(1+\frac{|x|}{k^2}\right) = \ln\left[\prod_{k\geqslant 1}\left(1-\frac{\left(i\sqrt{|x|}\right)^2}{k^2}\right)\right] = \ln\left[\frac{\sin\left(i\,\pi\sqrt{|x|}\right)}{i\,\pi\sqrt{|x|}}\right] \leqslant \ln\left[Ce^{\pi\sqrt{|x|}}\right].$$

The above inequalities for $(S_i)_{1\leqslant i\leqslant 3}$, together with (4.10), imply that there exists $C>0$ such that

$$\prod_{\lambda'\in\Sigma^H}\left(1+\frac{|x|}{|\lambda'|}\right) \leqslant Ce^{\pi\sqrt{|x|}}. \tag{4.11}$$

We pass now to evaluate the second product in (4.9). We have that

$$\prod_{\lambda'\in\Sigma^H\setminus\{\lambda\}}\left|\frac{\lambda'-\lambda}{\lambda'}\right| \geqslant \underbrace{\prod_{\lambda'\in\Sigma^H\setminus\{\lambda\}}\frac{|\Re(\lambda')-\Re(\lambda)|}{|\Re(\lambda')|}}_{P_1}\underbrace{\prod_{\lambda'\in\Sigma^H\setminus\{\lambda\}}\frac{|\Re(\lambda')|}{|\lambda'|}}_{P_2}.$$

We evaluate each of the products above. For the product P_1 we use property $P4)$ and the same arguments as in [12, Lemma 4.1]. We deduce that there exist two positive constants $B_0 > 0$ and $C > 0$ such that

$$(P_1)^{-1} \leqslant C\left(1 + |\Re(\lambda)|\right)^{B_0}.$$

For the second product, by using (4.2), we deduce that there exists $C > 0$ such that

$$(P_2)^2 = \prod_{\lambda' \in \Sigma^H \setminus \{\lambda\}} \frac{|\lambda'|^2 - (\Im(\lambda'))^2}{|\lambda'|^2} \geqslant \prod_{k \geqslant N_H} \left(1 - \frac{\delta^2}{(k^2-\delta)^2}\right) \geqslant C.$$

Hence, the last inequalities for $(P_i)_{1\leqslant i \leqslant 2}$ imply that there exist $B_0, C > 0$ such that

$$\prod_{\lambda' \in \Sigma^H \setminus \{\lambda\}} \left|\frac{\lambda'}{\lambda' - \lambda}\right| \leqslant C\left(1 + |\Re(\lambda)|\right)^{B_0}. \tag{4.12}$$

From (4.9), (4.11) and (4.12) it follows that (4.8) holds and the proof of the lemma is complete. □

Remark 4.3. *Estimate (4.7) implies that Φ_λ is a function of order not exceeding $\frac{1}{2}$ and a type not exceeding π if or order $\frac{1}{2}$. We recall that an entire function of exponential type $\tau \in (0,\infty)$ is a function of order not exceeding 1 and a type not exceeding τ if of order 1 (see [3, Page 8]).*

Let us define the $\mathcal{C}^\infty(\mathbb{R})$ function

$$\sigma_\nu := \begin{cases} \exp\left(-\frac{\nu}{1-t^2}\right) & \text{if } |t| < 1 \\ 0 & \text{if } |t| \geqslant 1, \end{cases} \tag{4.13}$$

where ν is a positive constant and put $c_\nu = 1/\|\sigma_\nu\|_{L^1}$. Given $\beta > 0$, we introduce the function

$$H_\beta(z) = c_\nu \int_{-1}^{1} \sigma_\nu(t) e^{-i\beta t z}\,\mathrm{d}t. \tag{4.14}$$

The properties of H_β we are interested in are listed in the following lemma.

Lemma 4.4. *Let $\beta > 0$, $\varrho > 0$ and set $\nu = (\pi + \varrho)^2/\beta$. The function H_β defined by (4.14) is an entire function of exponential type β which verifies the following properties:*

$$H_\beta(0) = 1, \tag{4.15}$$

$$|H_\beta(z)| \leqslant \exp(\beta|y|) \qquad (z = x + iy,\ x, y \in \mathbb{R}), \tag{4.16}$$

$$|H_\beta(x)| \leqslant C_3\sqrt{\nu+1}\exp\left(3\nu/4 - (\pi + \varrho/2)\sqrt{|x|}\right) \qquad (x \in \mathbb{R}), \tag{4.17}$$

$$|H_\beta(iy)| \geqslant \frac{1}{11\sqrt{\nu+1}}\exp\left(\beta|y|/(2\sqrt{\nu+1})\right) \qquad (y \in \mathbb{R}), \tag{4.18}$$

$$|H_\beta(x+iy)| \geqslant \frac{C_4}{\sqrt{\nu+1}}\exp\left(\beta|y|/(2\sqrt{\nu+1})\right) \qquad \left(y \in \mathbb{R}, \quad |x| \leqslant \frac{\pi}{4\beta}\right), \tag{4.19}$$

where C_3 and C_4 are two positive constants, independent of ϱ, ν and β.

Proof. From (4.14) we deduce that H_β, being the Fourier transform of a function in $L^2(-\beta,\beta)$, is an entire function of exponential type β. Properties (4.15)-(4.18) are proved in [12, Lemma 4.3] (see, also, [13, Lemma 9.2.3]). It remains to show estimate (4.19). We have that

$$|H_\beta(x+iy)| \geqslant c_\nu \left|\int_{-1}^{1} \sigma_\nu(t) e^{\beta t y}\cos(\beta t x)\,\mathrm{d}t\right|.$$

Since $0 \leqslant \beta|x| \leqslant \frac{\pi}{4}$, it follows that

$$\frac{\sqrt{2}}{2} \leqslant \cos(\beta t x) \leqslant 1,$$

from which we deduce that

$$|H_\beta(x+iy)| \geqslant c_\nu \frac{\sqrt{2}}{2} \int_{-1}^{1} \sigma_\nu(t) e^{\beta t y} \, \mathrm{d}t = \frac{\sqrt{2}}{2} H_\beta(iy). \tag{4.20}$$

By using (4.20) and (4.18) we deduce immediately that (4.19) holds and the proof of the lemma is complete. □

For each $\lambda \in \Sigma$, let us now define the entire function

$$Q_\lambda(z) = \Phi_\lambda(iz - \overline{\lambda}) \frac{H_\beta(z)}{H_\beta(-i\overline{\lambda})}. \tag{4.21}$$

The properties of the function Q_λ we are interested in are given by the following lemma.

Lemma 4.5. *For each $\lambda \in \Sigma$, the function Q_λ defined by* (4.21) *has the following properties:*

1. *For each $\lambda' \in \Sigma$ we have that* $Q_\lambda(-i\overline{\lambda'}) = \begin{cases} 1 & \text{if } \lambda = \lambda' \\ 0 & \text{if } \lambda \neq \lambda'. \end{cases}$
2. *Q_λ is an entire function of exponential type β.*
3. *There exists $\beta_0 > 0$ such that, for each $\beta \in (0, \beta_0]$, the following estimate holds*

$$|Q_\lambda(x)| \leqslant C_5 \left(1 + |\Re(\lambda)| + |x|\right)^{B_0} \exp\left(\frac{\kappa}{\beta} - \frac{\varrho}{2}\sqrt{|x|} - \beta|\Re(\lambda)|/(2\sqrt{\nu+1})\right) \quad (x \in \mathbb{R}), \tag{4.22}$$

where the two positive constants C_5 and κ are independent of $\beta \in (0, \beta_0]$.

Proof. From definitions (4.21) of the function Q_λ, (4.14) of the function H_β and (4.6) of the function Φ_λ we deduce immediately that the first property of Q_λ holds true. Moreover, by taking into account estimate (4.7) for Φ_λ and the fact that, according to Lemma 4.4, H_β is an entire function of exponential type β, it follows that Q_λ verifies the second property, too.

To prove (4.22), let us first recall that, according to (2.1), $|\Im(\lambda)| \leqslant D_0$ for each $\lambda \in \Sigma$. By taking $\beta_0 = \frac{\pi}{4D_0}$ we deduce from (4.19) that, for any $\beta \leqslant \beta_0$, the following estimate holds

$$\left|H_\beta(-i\overline{\lambda})\right| \geqslant \frac{C_4}{\sqrt{\nu+1}} \exp\left(\beta|\Re(\lambda)|/(2\sqrt{\nu+1})\right). \tag{4.23}$$

From (4.8), (4.17) and (4.23) we obtain that, for any $x \in \mathbb{R}$, we have

$$\begin{aligned} |Q_\lambda(x)| &= \left|\Phi_\lambda(ix - \overline{\lambda}) \frac{H_\beta(x)}{H_\beta(-i\overline{\lambda})}\right| \\ &\leqslant \frac{C_2 C_3}{C_4}(\nu+1)(1 + |\Re(\lambda)| + |x|)^{B_0} \exp\left(3\nu/4 - \varrho/2\sqrt{|x|} - \beta|\Re(\lambda)|/(2\sqrt{\nu+1})\right). \end{aligned} \tag{4.24}$$

Since $\nu = \frac{(\pi+\varrho)^2}{\beta}$, from (4.24) we deduce that the following estimate holds true for any $\beta \in (0, \beta_0]$

$$|Q_\lambda(x)| \leqslant \frac{C_5}{\beta}(1 + |\Re(\lambda)| + |x|)^{B_0} \exp\left(\frac{3(\pi+\varrho)^2}{4\beta} - \frac{\varrho}{2}\sqrt{|x|} - \beta|\Re(\lambda)|/(2\sqrt{\nu+1})\right) \qquad (x \in \mathbb{R}),$$

where C_5 is a positive constant independent of β. The last inequality gives (4.22) by taking $\kappa = \frac{3(\pi+\varrho)^2}{4} + 1$. □

We need to modify our function Q_λ given by (4.21) in order to be able to add conditions on the values of the derivatives.

Lemma 4.6. *Let $\lambda \in \Sigma$ and $\eta \in \mathbb{N}^*$. If Q_λ is the function defined by (4.21) then, for each $0 \leqslant j \leqslant \eta-1$, there exists a polynomial function $p_{\lambda,j}$ of degree less than or equal to $\eta-1$ such that the following properties are verified*

$$\forall k \leqslant \eta-1, \quad \left(Q_\lambda^\eta\, p_{\lambda,j}\right)^{(k)}(-i\overline{\lambda'}) = \begin{cases} 1 & \text{if } \lambda = \lambda' \text{ and } j = k \\ 0 & \text{otherwise.} \end{cases} \tag{4.25}$$

Moreover, there exists a positive constant C_6 independent of λ and $\beta \in (0, \beta_0)$, such that

$$|p_{\lambda,j}(z)| \leqslant \frac{C_6}{\beta^{\eta^2/2}} \left(1 + \left|z + i\overline{\lambda}\right|\right)^\eta e^{\eta^2 \beta |\Re(\lambda)|} \qquad (z \in \mathbb{C}). \tag{4.26}$$

Proof. We shall explicitly construct and evaluate the polynomials $p_{\lambda,j}$ for each $0 \leqslant j \leqslant \eta-1$. First, we define

$$p_{\lambda,\eta-1}(z) = \frac{1}{(\eta-1)!}(z + i\overline{\lambda})^{\eta-1}. \tag{4.27}$$

We remark that relations (4.25) are verified in the case $j = \eta-1$ and we have that

$$|p_{\lambda,\eta-1}(x)| \leqslant \frac{1}{(\eta-1)!}|z + i\overline{\lambda}|^{\eta-1}. \tag{4.28}$$

Generally, for $0 \leqslant j \leqslant \eta-1$, we define

$$p_{\lambda,j}(z) = \sum_{k=0}^{\eta-1} \frac{a_{\lambda,j,k}}{k!}(z + i\overline{\lambda})^k, \tag{4.29}$$

where the values $(a_{\lambda,j,k})_{0 \leqslant k \leqslant \eta-1}$ are given recursively as follows

$$\begin{cases} a_{\lambda,j,k} = 0 \qquad (0 \leqslant k < j) \\ \\ a_{\lambda,j,j} = 1, \\ \\ a_{\lambda,j,k} = -\displaystyle\sum_{s=1}^{k-j} C_k^s \left(Q_\lambda^\eta\right)^{(s)}(-i\overline{\lambda}) a_{\lambda,j,k-s} \qquad (j+1 \leqslant k \leqslant \eta-1). \end{cases} \tag{4.30}$$

Notice that (4.29) implies that $a_{\lambda,j,k} = p_{\lambda,j}^{(k)}(-i\overline{\lambda})$. From (4.30) it follows immediately that $Q_\lambda^\eta\, p_{\lambda,j}$ verifies (4.25). In order to prove (4.26), let us evaluate $\left(Q_\lambda^\eta\right)^{(s)}(-i\overline{\lambda})$ and show that there exists a constant $C > 0$ such that

$$\left|\left(Q_\lambda^\eta\right)^{(s)}(-i\overline{\lambda})\right| \leqslant \frac{C}{\beta^{\eta/2}} e^{\beta\eta|\Re(\lambda)|} \qquad (0 \leqslant s \leqslant \eta-1). \tag{4.31}$$

Indeed, by using the Cauchy formula, we have that

$$\left|\left(Q_\lambda^\eta\right)^{(s)}(-i\overline{\lambda})\right| = \left|\frac{s!}{2\pi\, i} \oint_{\Gamma_R} \frac{Q_\lambda^\eta(z)}{(z + i\overline{\lambda})^{s+1}}\, \mathrm{d}z\right|,$$

where $R \in (0, \gamma)$, $\Gamma_R = \{z \in \mathbb{C} \,:\, |z + i\overline{\lambda}| = R\}$ and the contour integral is taken counter-clockwise.

Let us evaluate $Q_\lambda(z)$ on Γ_R. From (4.21), (4.7), (4.16) and (4.19) it follows that, for any $z \in \Gamma_R$ of the form $z = -i\overline{\lambda} + Re^{iv}$, $v \in [0, 2\pi]$, we have that

$$|Q_\lambda(z)| = \left|\Phi_\lambda(iRe^{iv}) \frac{H_\beta(-i\overline{\lambda} + Re^{iv})}{H_\beta(-i\overline{\lambda})}\right| \leqslant \frac{C_1}{C_4}(1+R)^{B_0}\sqrt{\nu+1}e^{\pi\sqrt{R} + R\beta + \beta|\Re(\lambda)|},$$

and (4.31) is proved.

Now, from (4.30) and (4.31), we immediately deduce that

$$\left|p_{\lambda,j}^{(k)}(-i\overline{\lambda})\right| \leqslant \frac{C}{\beta\eta^2/2} e^{\beta\eta(k-j)|\Re(\lambda)|} \qquad (k \geqslant j),$$

which, by taking into account (4.29), implies that

$$|p_{\lambda,j}(z)| \leqslant \frac{C}{\beta\eta^2/2}|z+i\overline{\lambda}|^j \sum_{s=0}^{\eta-1-j} \frac{1}{(s+j)!}|z+i\overline{\lambda}|^s e^{\beta\eta s|\Re(\lambda)|} \qquad (z \in \mathbb{C}). \tag{4.32}$$

Estimate (4.32) shows that (4.26) holds and the proof of the lemma is complete. □

For each $\lambda \in \Sigma$ and $j \in \{0, 1, ..., \eta - 1\}$, let us define the function

$$G_{\lambda,j}(z) = Q_\lambda^\eta(z) p_{\lambda,j}(z) \qquad (z \in \mathbb{C}), \tag{4.33}$$

where Q_λ is given by (4.21) and $p_{\lambda,j}$ are the polynomials functions from Lemma 4.6. The following theorem studies the main properties of the function $G_{\lambda,j}$.

Theorem 4.7. *For each $\lambda \in \Sigma$ and $j \in \{0, 1, ..., \eta - 1\}$, the function $G_{\lambda,j}$ defined by (4.33) has the following properties:*

1. *For each $\lambda' \in \Sigma$ we have that*

$$\left(G_{\lambda,j}\right)^{(k)}(-i\overline{\lambda'}) = \begin{cases} 1 & \text{if } \lambda = \lambda' \text{ and } j = k \\ 0 & \text{otherwise.} \end{cases} \qquad (1 \leqslant k \leqslant \eta - 1). \tag{4.34}$$

2. *$G_{\lambda,j}$ is an entire function of exponential type $\eta\beta$.*
3. *There exists $\beta_0 > 0$ such that, for each $\beta \in (0, \beta_0]$, the following estimate holds*

$$\|G_{\lambda,j}\|_{L^2(\mathbb{R})} \leqslant \frac{C_7}{|\Re(\lambda)|+1} \exp\left(\frac{\kappa}{\beta} + \eta^2\beta|\Re(\lambda)|\right), \tag{4.35}$$

where the two positive constants C_7 and κ are independent of λ, j and $\beta \in (0, \beta_0]$.

Proof. Properties (4.34) are consequences of (4.25). Since the polynomials $p_{\lambda,j}$ have degree less than η and Q_λ is an entire function of exponential type β, it follows that $G_{\lambda,j}$ is an entire function of exponential type $\eta\beta$. Finally, estimate (4.35) is a consequence of estimates (4.22) and (4.26).

Indeed from (4.22) and (4.26), we have that

$$\begin{aligned} |G_{\lambda,j}(x)|^2 &= |Q_\lambda^\eta(x) p_{\lambda,j}(x)|^2 \\ &\leqslant \frac{(C_5C_6)^2}{\beta\eta^2}(1+|\Re(\lambda)|+|x|)^{2B_0} \exp\left(\frac{2\kappa}{\beta} - \varrho\sqrt{|x|} - \beta|\Re(\lambda)|/\sqrt{\nu+1}\right)\left(1+\left|x+i\overline{\lambda}\right|\right)^{2\eta} e^{2\eta^2\beta|\Re(\lambda)|} \\ &\leqslant \frac{(C_5C_6)^2}{\beta\eta^2} \exp\left(\frac{2\kappa}{\beta} + 2\eta^2\beta|\Re(\lambda)|\right) c'' (1+|\Re(\lambda)|+|x|)^{2B_0+2\eta} \exp\left(-\varrho\sqrt{|x|} - \beta|\Re(\lambda)|/\sqrt{\nu+1}\right). \end{aligned}$$

Now, we have that

$$\begin{aligned} &(1+|\Re(\lambda)|+|x|)^{2B_0+2\eta} \exp\left(-\varrho\sqrt{|x|} - \beta|\Re(\lambda)|/\sqrt{\nu+1}\right) \\ &\leqslant (1+|\Re(\lambda)|)^{2B_0+2\eta} \exp\left(-\beta|\Re(\lambda)|/\sqrt{\nu+1}\right)(1+|x|)^{2B_0+2\eta} \exp\left(-\varrho\sqrt{|x|}\right) \\ &\leqslant C\left(\frac{\sqrt{\nu+1}}{\beta}\right)^{2B_0+2\eta} \frac{1}{(1+|\Re(\lambda)|)^2}(1+|x|)^{2B_0+2\eta} \exp\left(-\varrho\sqrt{|x|}\right), \end{aligned}$$

where we have used in the last inequality that $(1+s)^{2B_0+2\eta+2}e^{-s} \leqslant C = C(\eta, B_0)$ for any $s \geqslant 0$. The conclusion follows by taking into account that

$$\int_{\mathbb{R}} (1+|x|)^{2B_0+2\eta} \exp\left(-\varrho\sqrt{|x|}\right) \mathrm{d}x \leqslant C = C(\eta, B_0, \varrho)$$

and, for any $\kappa' > \kappa$,

$$\frac{1}{\beta^{\eta^2}} \exp\left(\frac{2\kappa}{\beta} + 2\eta^2\beta|\Re(\lambda)|\right)\left(\frac{\sqrt{\nu+1}}{\beta}\right)^{2B_0+2\eta} \leqslant C\exp\left(\frac{2\kappa'}{\beta} + 2\eta^2\beta|\Re(\lambda)|\right).$$

□

Now we have all the ingredients needed to construct our biorthogonal family. For each $\lambda \in \Sigma$ and $j \in \{0, 1, ..., \eta-1\}$, we define the function $\theta_{\lambda,j}$ as follows

$$\theta_{\lambda,j}(t) = \frac{(-i)^j}{2\pi}\int_{\mathbb{R}} G_{\lambda,j}(x)e^{ixt}\,\mathrm{d}x. \tag{4.36}$$

The following result is a consequence of the properties of the function $G_{\lambda,j}$ proved in Theorem 4.7.

Theorem 4.8. *Let Σ be the family of eigenvalues given by (3.3) and let $\eta \in \mathbb{N}^*$. There exists $T_0 > 0$ such that, for any $T \in (0, T_0)$, the family of functions $\Lambda = \left(t^j e^{-\lambda t}\right)_{\lambda\in\Sigma,\, 0\leqslant j\leqslant \eta-1}$ has a biorthogonal $(\theta_{\lambda,j})_{\lambda\in\Sigma,\, 0\leqslant j\leqslant\eta-1}$ in $L^2\left(-\frac{T}{2}, \frac{T}{2}\right)$ with the property that*

$$\|\theta_{\lambda,j}\|_{L^2\left(-\frac{T}{2},\frac{T}{2}\right)} \leqslant \frac{c}{|\Re(\lambda)|+1}\exp\left(\frac{2\eta^2\kappa}{T} + \frac{T}{2}|\Re(\lambda)|\right), \tag{4.37}$$

where the two positive constants c and κ are independent of λ, j and T.

Proof. Let us chose $T_0 = \min\{2\eta\beta_0, 1\}$, $T \in (0, T_0)$ and $\beta = \frac{T}{2\eta^2}$. The Paley-Wiener Theorem and Theorem 4.7 imply that (4.36) defines a function $\theta_{\lambda,j}$ which belongs to $L^2\left(-\frac{T}{2}, \frac{T}{2}\right)$. Moreover, since

$$G_{\lambda,j}(z) = \frac{1}{(-i)^j}\int_{-\frac{T}{2}}^{\frac{T}{2}} \theta_{\lambda,j}(t)e^{-izt}\,\mathrm{d}t \qquad (z \in \mathbb{C}),$$

relations (4.34) imply that $(\theta_{\lambda,j})_{\lambda\in\Sigma,\,0\leqslant j\leqslant\eta-1}$ is a biorthogonal family to Λ in $L^2\left(-\frac{T}{2}, \frac{T}{2}\right)$. Since the Plancherel formula implies that $\sqrt{2\pi}\|\theta_{\lambda,j}\|_{L^2\left(-\frac{T}{2},\frac{T}{2}\right)} = \|G_{\lambda,j}\|_{L^2(\mathbb{R})}$, from estimate (4.35) we deduce that (4.37) is verified with $c = \frac{C_7}{\sqrt{2\pi}}$ which completes the theorem's proof. □

The following result is a consequence of Theorem 4.8.

Corollary 4.9. *Under the hypothesis of Theorem 4.8 the following inequality holds true*

$$\sum_{\lambda\in\Sigma,\,0\leqslant j\leqslant\eta-1} |a_{\lambda,j}|^2 e^{-2T\Re(\lambda)} \leqslant c\exp\left(\frac{4\eta^2\kappa}{T}\right)\int_0^T \left|\sum_{\lambda\in\Sigma,\,0\leqslant j\leqslant\eta-1} a_{\lambda,j}t^j e^{-\lambda t}\right|^2 \mathrm{d}t \tag{4.38}$$

for any finite sequence of complex numbers $(a_{\lambda,j})_{\lambda\in\Sigma,\,0\leqslant j\leqslant\eta-1}$.

Proof. Let $(\theta_{\lambda,j})_{\lambda\in\Sigma,\,0\leqslant j\leqslant\eta-1}$ be the biorthogonal to Λ in $L^2\left(-\frac{T}{2}, \frac{T}{2}\right)$ given by Theorem 4.8. It follows that there exists a biorthogonal $\left(\widehat{\theta}_{\lambda,j}\right)_{\lambda\in\Sigma,\,0\leqslant j\leqslant\eta-1}$ to Λ in $L^2(0,T)$ such that

$$\|\widehat{\theta}_{\lambda,j}\|_{L^2(0,T)} \leqslant \frac{c}{|\Re(\lambda)|+1}\exp\left(\frac{2\eta^2\kappa}{T} + T|\Re(\lambda)|\right). \tag{4.39}$$

Indeed, $\left(\widehat{\theta}_{\lambda,j}\right)_{\lambda\in\Sigma,\,0\leqslant j\leqslant \eta-1}$ is given by the following formulas

$$\widehat{\theta}_{\lambda,j}\left(t+\frac{T}{2}\right)=\left(\theta_{\lambda,j}(t)+\sum_{s=j+1}^{\eta-1} l_{j,s}\theta_{\lambda,s}(t)\right)e^{\frac{T}{2}\lambda}\qquad \left(t\in\left(-\frac{T}{2},\frac{T}{2}\right)\right),\tag{4.40}$$

where the coefficients $l_{j,s}$ can be obtained recursively

$$\begin{cases} l_{j,j+1}=-C_{j+1}^{j}\left(\frac{T}{2}\right)^{j}, \\ l_{j,s}=-C_{s}^{j}\left(\frac{T}{2}\right)^{j}-\sum_{r=1}^{s-j-1}C_{s}^{r}\left(\frac{T}{2}\right)^{r}l_{j,s-r}, \qquad j+2\leqslant s\leqslant \eta-1.\end{cases}\tag{4.41}$$

From (4.37), (4.40) and (4.41) it follows that (4.39) holds with a constant c depending only on T_0 and η.

Let us now pass to prove (4.38). By using the orthogonality properties of $(\theta_{\lambda,j})_{\lambda\in\Sigma,\,0\leqslant j\leqslant\eta-1}$ and Cauchy inequality we obtain that

$$\sum_{\lambda\in\Sigma,\,0\leqslant j\leqslant\eta-1}|a_{\lambda,j}|^2e^{-2T\Re(\lambda)}=$$

$$=\int_0^T\left(\sum_{\lambda\in\Sigma,\,0\leqslant j\leqslant\eta-1}a_{\lambda,j}\,e^{-2T\Re(\lambda)}\widehat{\theta}_{\lambda,j}(t)\right)\overline{\left(\sum_{\lambda\in\Sigma,\,0\leqslant j\leqslant\eta-1}a_{\lambda,j}\,t^je^{-\lambda t}\right)}\,\mathrm{d}t\leqslant$$

$$\leqslant\left(\int_0^T\left|\sum_{\lambda\in\Sigma,\,0\leqslant j\leqslant\eta-1}a_{\lambda,j}\,e^{-2T\Re(\lambda)}\widehat{\theta}_{\lambda,j}(t)\right|^2\mathrm{d}t\right)^{1/2}\left(\int_0^T\left|\sum_{\lambda\in\Sigma,\,0\leqslant j\leqslant\eta-1}a_{\lambda,j}\,t^je^{-\lambda t}\right|^2\mathrm{d}t\right)^{1/2}\leqslant$$

$$\leqslant\left(\sum_{\lambda\in\Sigma,\,0\leqslant j\leqslant\eta-1}|a_{\lambda,j}|^2e^{-2T\Re(\lambda)}\right)^{1/2}\left(\sum_{\lambda\in\Sigma,\,0\leqslant j\leqslant\eta-1}e^{-2T\Re(\lambda)}\|\widehat{\theta}_{\lambda,j}\|^2_{L^2(0,T)}\right)^{1/2}$$

$$\left(\int_0^T\left|\sum_{\lambda\in\Sigma,\,0\leqslant j\leqslant\eta-1}a_{\lambda,j}\,t^je^{-\lambda t}\right|^2\mathrm{d}t\right)^{1/2}.$$

By taking into account estimates (4.39) and since (4.5) implies that

$$\sum_{\lambda\in\Sigma}\frac{1}{(|\Re(\lambda)|+1)^2}<\infty,$$

it follows that (4.38) holds and the proof ends. □

5 Observability result

Now, we have all the ingredients needed to prove the observability inequality (1.13) for the solutions of the adjoint equation (1.10).

Proof of Theorem 1.2: Let T_0 be given by Theorem 4.8 and let $T\in(0,T_0)$. We recall that the set $\left(\widetilde{\psi}_k\right)_{1\leqslant k\leqslant N_L}\cup(\psi_k)_{k\geqslant N_H}$ is a Riesz basis in $L^2(0,\pi)$ (see Theorem 3.1) and, if $\varphi_0\in L^2(0,\pi)$ is given by (3.5) then the corresponding solution φ of (1.10) is given by (3.6). Consequently, we have that

$$
\begin{aligned}
\|\varphi(T)\|^2_{L^2(0,\pi)} \leqslant M_1 \Bigg[\sum_{k=N_H}^{\infty} |a_k|^2 e^{-2\Re(\lambda_k)T} + \sum_{k=1}^{N_L}\sum_{j=1}^{d_k} \Bigg(\left|a_{k,j,r_{k,j}-1}\right|^2 + \left|a_{k,j,r_{k,j}-2} + \frac{T}{1!} a_{k,j,r_{k,j}-1}\right|^2 \\
+ \dots + \left| a_{k,j,0} + \frac{T}{1!} a_{k,j,1} + \frac{T^2}{2!} a_{k,j,2} + \dots + \frac{T^{r_{k,j}-1}}{(r_{k,j}-1)!} a_{k,j,r_{k,j}-1}\right|^2 \Bigg) e^{-2\Re(\lambda_k)T}\Bigg]. \qquad (5.1)
\end{aligned}
$$

Since $T < T_0 \leqslant 1$, from (5.1) we deduce that

$$
\|\varphi(T)\|^2_{L^2(0,\pi)} \leqslant M_1 \left(\sum_{k=1}^{N_L}\sum_{j=1}^{d_k}\sum_{m=0}^{r_{k,j}-1} \left|a_{k,j,m}\right|^2 e^{-2\Re(\lambda_k)T} + \sum_{k=N_H}^{\infty} |a_k|^2 e^{-2\Re(\lambda_k)T} \right), \qquad (5.2)
$$

where M_1 is a positive constant independent of T. Let us now evaluate the right hand side of (1.13). To this aim, we use the solution $\varphi(t)$ of (1.10) written in the form (3.9). From (4.38) we obtain that

$$
\begin{aligned}
\int_0^T \int_\omega |\varphi(t,x)|^2 \,\mathrm{d}x\,\mathrm{d}t &= \int_\omega \int_0^T |\varphi(t,x)|^2 \,\mathrm{d}t\,\mathrm{d}x \qquad (5.3) \\
&\geqslant c e^{-\frac{4\eta^2\kappa}{T}} \left(\sum_{k=1}^{N_L} e^{-2T\Re(\lambda_k)} \sum_{s=0}^{R_k-1} \int_\omega |S_{k,s}(x)|^2 \,\mathrm{d}x + \sum_{k=N_H}^{\infty} |a_k|^2 e^{-2T\Re(\lambda_k)} \int_\omega |\psi_k(x)|^2\,\mathrm{d}x \right),
\end{aligned}
$$

where

$$
S_{k,s}(x) = \sum_{p=0}^{R_k-s-1} \sum_{m=s}^{R_k-p-1} \sum_{j\in\mathcal{I}_{k,p}} \frac{1}{s!} a_{k,j,m}\psi_{k,j,m-s}(x). \qquad (5.4)
$$

For each $1 \leqslant k \leqslant N_L$, let $\mathcal{S}_k$ be the root space corresponding to the eigenvalues λ_k of dimension $N_P(k) = \sum_{j=1}^{d_k} r_{kj}$. For each $1 \leqslant k \leqslant N_L$, the following two properties hold true:

(P1) If $S_{k,s|_\omega} \equiv 0$ for $0 \leqslant s \leqslant R_k - 1$, then

$$
a_{k,j,m} = 0 \qquad (1 \leqslant j \leqslant d_k,\ 0 \leqslant m \leqslant r_{kj}-1). \qquad (5.5)
$$

(P2) For each $1 \leqslant k \leqslant N_L$, the map

$$
\mathbb{C}^{N_P(k)} \ni (a_{k,j,m})_{1\leqslant j\leqslant d_k,\ 0\leqslant m\leqslant r_{kj}-1} \mapsto \left(\sum_{s=0}^{R_k-1} \int_\omega |S_{k,s}(x)|^2\,\mathrm{d}x \right)^{1/2} \in \mathbb{R}_+, \qquad (5.6)
$$

is a norm in $\mathbb{C}^{N_P(k)}$.

Property $(P2)$ is a direct consequence of definition (5.4) and property $(P1)$. To prove property $(P1)$ we use the unique continuation principle given by Lemma 2.15. Notice that the properties of our kernel K allow us to apply this Lemma. Since

$$
S_{k,R_k-1}(x) = \frac{1}{(R_k-1)!} \sum_{j\in\mathcal{I}_{k,0}} a_{k,j,0}\psi_{k,j,0}(x),
$$

and $\psi_{k,j,0} \in \ker(A^* - \lambda_k \mathcal{I})$, $j \in \mathcal{I}_{k,0}$, from Lemma 2.15 we deduce that

$$
a_{k,j,R_k-1} = 0 \qquad (j \in \mathcal{I}_{k,0}). \qquad (5.7)
$$

Since

$$S_{k,R_k-2}(x) = \frac{1}{(R_k-2)!}\left(\sum_{j\in\mathcal{I}_{k,0}} a_{k,j,R_k-2}\psi_{k,j,0}(x) + \sum_{j\in\mathcal{I}_{k,0}} a_{k,j,R_k-1}\psi_{k,j,1}(x) \right.$$
$$\left. + \sum_{j\in\mathcal{I}_{k,1}} a_{k,j,R_k-2}\psi_{k,j,0}(x)\right),$$

and $\psi_{k,j,0} \in \ker(A^* - \lambda_k\mathcal{I})$, $j \in \mathcal{I}_{k,0} \cup \mathcal{I}_{k,1}$, from (5.7) and Lemma 2.15 we deduce that

$$a_{k,j,R_k-2} = 0 \qquad (j \in \mathcal{I}_{k,0} \cup \mathcal{I}_{k,1}). \tag{5.8}$$

If we suppose that, for some $0 \leqslant q \leqslant R_k - 1$, we have that

$$a_{k,j,l} = 0 \qquad (j \in \cup_{s=0}^{R_k-2-q}\mathcal{I}_{k,s}, \quad q+1 \leqslant l \leqslant R_k - 1), \tag{5.9}$$

then it follows that

$$S_{k,q}(x) = \frac{1}{q!}\sum_{p=0}^{R_k-q-1}\sum_{j\in\mathcal{I}_{k,p}} a_{k,j,q}\psi_{k,j,0}(x).$$

Since $\psi_{k,j,0} \in \ker(A^* - \lambda_k\mathcal{I})$ for $j \in \cup_{p=0}^{R_k-q-1}\mathcal{I}_{k,p}$, by using again Lemma 2.15 we deduce that

$$a_{k,j,q} = 0 \qquad (j \in \cup_{l=0}^{R_k-1-q}\mathcal{I}_{k,l}). \tag{5.10}$$

Hence, we have proved by induction that $(P1)$ holds true.

From $(P2)$ we deduce that, for each $1 \leqslant k \leqslant N_L$, there exists a constant $c_k > 0$ such that,

$$\sum_{s=0}^{R_k-1}\int_\omega |S_{k,s}(x)|^2\,\mathrm{d}x \geqslant c_k\sum_{j=1}^{d_k}\sum_{m=0}^{r_{kj}-1}|a_{k,j,m}|^2. \tag{5.11}$$

From (5.11), by taking $c' = \min_{1\leqslant k\leqslant N_L} c_k$, we deduce that

$$\sum_{k=1}^{N_L} e^{-2T\Re(\lambda_k)}\sum_{s=0}^{R_k-1}\int_\omega |S_{k,s}(x)|^2\,\mathrm{d}x \geqslant c'\sum_{k=1}^{N_L}\sum_{j=1}^{d_k}\sum_{m=0}^{r_{kj}-1}|a_{k,j,m}|^2 e^{-2T\Re(\lambda_k)}. \tag{5.12}$$

Using (2.47), we have that

$$\sum_{k=N_H}^{\infty}|a_k|^2 e^{-2T\Re(\lambda_k)}\int_\omega|\psi_k(x)|^2\,\mathrm{d}x \geqslant c''\sum_{k=N_H}^{\infty}|a_k|^2 e^{-2T\Re(\lambda_k)}. \tag{5.13}$$

From (5.3), (5.12) and (5.13) it follows that

$$\int_0^T\int_\omega|\varphi(t,x)|^2\,\mathrm{d}x\,\mathrm{d}t$$
$$\geqslant M_2 e^{-\frac{4\eta^2\kappa}{T}}\left(\sum_{k=1}^{N_L}\sum_{j=1}^{d_k}\sum_{m=0}^{r_{kj}-1}|a_{k,j,m}|^2 e^{-2T\Re(\lambda_k)} + \sum_{k=N_H}^{\infty}|a_k|^2 e^{-2T\Re(\lambda_k)}\right), \tag{5.14}$$

where $M_2 = c\min\{c', c''\}$ is a positive constant independent of T. By taking into account (5.2) and (5.14), we deduce that (1.13) holds with $M_0 = \frac{M_1}{M_2}$ and $\varsigma = 4\eta^2\kappa$ which concludes the proof of the theorem. □

6 The nonlinear problem

The aim of this section is to provide the proof of Theorem 1.3. Without loss of generality, we may suppose that $T < T_0$, where T_0 is given in Theorem 1.2. Let us consider that

$$K(\xi, x) = \partial_{xx} w^S(\xi) e^{W(\xi)} \partial_{xx} w^S(x) e^{-W(x)},$$

with $W(x) = \frac{1}{2}\int_0^x w^S \, ds$. Since $\partial_{xx} w^S \not\equiv 0$ in ω, the hypotheses Theorem 1.2 are satisfied and (1.13) holds true.

Let $X = L^2(0,\pi)$ and $U = L^2(\omega)$ and let us denote

$$\gamma : (0,\infty) \to [0,\infty), \quad t \mapsto \gamma(t) := M_0 \exp\left(\frac{\varsigma}{t}\right),$$

where M_0 and ς are the constants in (1.13).

As stated in Theorem 1.1, for any $z_0 \in X$, there exists $v \in L^2(0,T;U)$ such that the solution of (1.8) satisfies

$$z(T) = 0$$

and

$$\|v\|_{L^2(0,T;U)} \leqslant \gamma(T)\|z_0\|_X.$$

By using this result, one can handle the controllability of a system similar to (1.8) but with right-hand side (see [10]). We introduce here some notation in order to state such a result.

Let $r \in (1,2)$ be a constant. We consider ς_0, ς_1, ς_* such that

$$\varsigma_1 \left(\frac{1}{r^2} - \frac{1}{4}\right) > \frac{\varsigma}{r-1}, \quad \varsigma_0 = \frac{\varsigma_1}{r^2} - \frac{\varsigma}{r-1}, \quad \varsigma_0 > \varsigma_* > \frac{\varsigma_1}{4} > 0. \tag{6.1}$$

We define three functions in $(0,T)$ by

$$\rho_{\mathcal{F}}(t) = \exp\left(-\frac{\varsigma_1}{T-t}\right), \quad \rho_0(t) = M_0 \exp\left(-\frac{\varsigma_0}{T-t}\right), \quad \rho(t) = \exp\left(-\frac{\varsigma_*}{T-t}\right), \tag{6.2}$$

so that $\rho_{\mathcal{F}}$, ρ_0 and ρ are continuous, decreasing and $\rho_{\mathcal{F}}(T) = \rho_0(T) = \rho(T) = 0$. We associate to the functions $\rho_{\mathcal{F}}$ and ρ_0 the Hilbert spaces $\mathcal{F}$ and $\mathcal{U}$ defined by

$$\mathcal{F} = \left\{ f \in L^2(0,T;X) \;\middle|\; \frac{f}{\rho_{\mathcal{F}}} \in L^2(0,T;X) \right\}, \tag{6.3}$$

$$\mathcal{U} = \left\{ u \in L^2(0,T;U) \;\middle|\; \frac{u}{\rho_0} \in L^2(0,T;U) \right\}. \tag{6.4}$$

By supposing that $z_0 \in D((-A)^{\frac{1}{2}})$ and $f \in \mathcal{F}$, we are now able to obtain a controllability result for the nonhomogeneous equation

$$\begin{cases} \partial_t z(t) + Az(t) = f + v\chi_\omega, \\ z(0) = z_0. \end{cases} \tag{6.5}$$

In order to do this we use the controllability of (1.8) from Theorem 1.2 and the fact that $\rho_{\mathcal{F}}$, ρ_0 and ρ satisfy the following relations

$$\rho_0(t) = \rho_{\mathcal{F}}(r^2(t-T)+T)\gamma((r-1)(T-t)) \quad \left(t \in \left[T\left(1-\frac{1}{r^2}\right), T\right]\right),$$

$$\frac{\rho_0}{\rho}, \; \frac{\rho_{\mathcal{F}}}{\rho}, \; \frac{\rho' \rho_0}{\rho^2} \in L^\infty(0,T).$$

Consequently, applying Proposition 2.3 and Proposition 2.8 of [10], we deduce that for every $z_0 \in D((-A)^{\frac{1}{2}})$ and for every $f \in \mathcal{F}$, there exists $v \in \mathcal{U}$ such that the solution z of (6.5) satisfies

$$\frac{z}{\rho} \in L^2(0,T;D(A)) \cap H^1(0,T;X) \cap C([0,T];D((-A)^{\frac{1}{2}})).$$

Moreover, there exists a positive constant C such that

$$\left\| \frac{z}{\rho} \right\|_{L^2(0,T;D(A))\cap H^1(0,T;X)\cap C([0,T];D((-A)^{\frac{1}{2}}))} \leqslant C \left(\|z_0\|_{D((-A)^{1/2})} + \|f\|_{\mathcal{F}} \right). \tag{6.6}$$

Using the change of variables (1.5) and definition (1.7) of A, we deduce that for every $y_0 \in H_0^1(0,\pi)$, and for every $f \in \mathcal{F}$, there exists $u \in \mathcal{U}$ such that the solution y of

$$\begin{cases} \partial_t y - \nu^S \partial_{xx} y - \mu \left(\int_0^\pi (\partial_x w^S)(\partial_x y)\ dx \right) \partial_{xx} w^S + w^S \partial_x y + y \partial_x w^S = f + u\chi_\omega, \\ y(t,0) = y(t,\pi) = 0, \\ y(0,\cdot) = y_0, \end{cases} \tag{6.7}$$

satisfies

$$\frac{y}{\rho} \in L^2(0,T;H^2(0,\pi)) \cap H^1(0,T;L^2(0,\pi)) \cap C([0,T];H_0^1(0,\pi)),$$

with the estimate

$$\left\| \frac{y}{\rho} \right\|_{L^2(0,T;H^2(0,\pi))\cap H^1(0,T;L^2(0,\pi))\cap C([0,T];H_0^1(0,\pi))} \leqslant C \left(\|y_0\|_{H_0^1(0,\pi)} + \|f\|_{\mathcal{F}} \right). \tag{6.8}$$

Notice that (6.8) implies, in particular, that

$$y(T) = 0. \tag{6.9}$$

In order to prove Theorem 1.3, we only need to show that the mapping

$$\mathcal{N} : f \in \mathcal{F} \to F(y) \in \mathcal{F},$$

where F is given by (1.3), is well-defined and admits a fixed point.

In order to do this, we first notice that (6.1) and (6.2) yield

$$\frac{\rho^4}{\rho_{\mathcal{F}}} \leqslant C. \tag{6.10}$$

Using this inequality, relation (6.8) and standard properties of Sobolev spaces we find that

$$\|F(y)\|_{\mathcal{F}} \leqslant C \left(\|y_0\|_{H_0^1(0,\pi)} + \|f\|_{\mathcal{F}} \right)^2 \left(1 + \left(\|y_0\|_{H_0^1(0,\pi)} + \|f\|_{\mathcal{F}} \right)^2 \right). \tag{6.11}$$

Let us consider f^1 and f^2 in $\mathcal{F}$. Assume that y^1 and y^2 are the solutions of (6.7) associated with f^1 and f^2, respectively, with the initial condition y_0 and satisfying (6.8). Then some calculation and (6.10) imply

$$\begin{aligned} \|F(y^1) - F(y^2)\|_{\mathcal{F}} \leqslant C\|f^1 - f^2\|_{\mathcal{F}} \Big(& \left(\|y_0\|_{H_0^1(0,\pi)} + \|f^1\|_{\mathcal{F}} + \|f^2\|_{\mathcal{F}} \right) \\ & + \left(\|y_0\|_{H_0^1(0,\pi)} + \|f^1\|_{\mathcal{F}} + \|f^2\|_{\mathcal{F}} \right)^3 \Big). \end{aligned} \tag{6.12}$$

Estimates (6.11)-(6.12) imply that we can use the Banach fixed point theorem on a ball of $\mathcal{F}$ of radius $\|y_0\|_{H_0^1(0,\pi)}$. If $\|y_0\|_{H_0^1(0,\pi)}$ is small enough, then the above estimates yield that F is a contraction and this completes the proof of the theorem.

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